Misperceived Law of Motion in Macroeconomic Expectations [*]

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Abstract

This paper offers a new empirical method to detect discrepancies between perceived and actual laws of motion across a broad range of expectation formation models. This method hinges on a theoretical observation: when perceived and actual laws of motion align, the cross-covariance between consensus expectations reported at the same period and the corresponding outcomes forms a symmetric matrix. By selecting expectations of different variables and their forecasts over multiple horizons, one can detect various types of discrepancies. Applications of this method to the MSC and SPF data support under-extrapolation and cognitive discounting, but not reduced model complexity.

Keywords: Misperceived Law of Motion, Survey Expectations, Decisive Evidence

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1 Introduction

Dynamic macroeconomic models typically involve both an actual law of motion that governs the evolution of macroeconomic indicators over time and a perceived law of motion that represents agents' perceptions of the actual law of motion. When discrepancies arise between the perceived and actual laws of motion, they give rise to a misperceived law of motion. Despite the growing body of literature exploring the macroeconomic implications of this misperception,[1] existing empirical methods are primarily designed to estimate specific models rather than providing direct evidence for this misperception.[2] In contrast, the objective of my paper is to develop an empirical method to detect misperceived law of motion across a wide range of linear Gaussian models. This method aims to directly select the plausible models that account for the empirical evidence in survey expectations.

The method is built upon a theoretical observation regarding the cross-covariance matrix of the consensus expectations reported contemporaneously and the corresponding outcomes. In general, the cross-covariance matrix is not symmetric. However, when perceived and actual laws of motion align, the cross-covariance matrix becomes symmetric. This theoretical observation serves as the empirical basis for detecting misperceived law of motion. Furthermore, by selecting the expectations of different variables and their forecasts across multiple horizons, one can detect various types of misperceived laws of motion.

For instance, if the vector of expectations includes the nowcast and forecast of growth, one can detect the misperceived law of motion by comparing the covariance of the growth forecast and the current growth with the covariance of the growth nowcast and the forward growth. Moreover, the ratio between these two covariance statistics can be interpreted as the degree of under- or over-extrapolation.

[1] e.g. Eusepi and Preston (2011); Farhi and Werning (2019); Gabaix (2020); Molavi (2022).

[2] e.g. Angeletos, Huo and Sastry (2020); Kučinskas and Peters (2023); Farmer, Nakamura and Steinsson (2023); Xie (2023).

When is the method valid? The method is valid within a general linear Gaussian framework that consists of two fundamental components: an actual law of motion and a signal extraction problem. The actual law of motion governs the evolution of indicators over time and is represented by a linear Gaussian stationary state-space model. The signal extraction problem pertains to how forecasters form expectations, which involves a linear projection of outcomes onto the observed noisy signals. Notably, in this framework, the linear projection is computed based on the perceived law of motion and the perceived signal precision, rather than their actual counterparts. Additionally, the explicit structure of this framework does not need to be specified, so that the same method can be applied to all models within the general framework.

This framework has the flexibility to encompass a board range of misperceptions. One type of the misperception emphasizes information rigidities, where agents face limitations in accessing or updating information.[3] Another type focuses on misperceived signal precision, which represents deviations from rational expectations as if there are errors in assessing the quality of received signals.[4] The third type highlights misperceived law of motion, where agents' perceptions of the actual law of motion differ from reality.[5] Notably, in the special case with an AR(1) law of motion, my framework nests the framework of Angeletos et al. (2020), which combines the three types of misperceptions above in a unified model.

[3]Examples include sticky information (e.g. Mankiw and Reis, 2002), noisy information (e.g. Woodford, 2003; Angeletos and La'O, 2013), and rational inattention (e.g. Sims, 2003; Maćkowiak and Wiederholt, 2009).

[4]Examples include overconfidence (e.g. Daniel, Hirshleifer and Subrahmanyam, 1998; Broer and Kohlhas, 2023), diagnostic expectations (e.g. Bordalo, Gennaioli and Shleifer, 2018; Bordalo, Gennaioli, Ma and Shleifer, 2020), and noisy ambiguity aversion (e.g. Huo, Pedroni and Pei, 2023).

[5]Examples include level-k thinking (e.g. Farhi and Werning, 2019; Qiu, 2019), cognitive discounting (e.g. Gabaix, 2020), and learning via oversimplified models (e.g. Molavi, 2022).

To focus on detecting misperceived law of motion, the framework in this paper relies on two restrictions. Firstly, it assumes a linear Gaussian structure, which excludes nonlinear models such as learning about the law of motion[6] and regime switch.[7] Secondly, the subjective expectations are derived from a signal extraction problem, making the framework less general compared with Kučinskas and Peters (2023). When these two restrictions are removed, the new method proposed in this paper only detects deviations from rational expectations, which is a less precise prediction compared to misperceived law of motion.

Why does the method work? When both the perceived and actual laws of motion follow an AR(1) process, a method proposed by Reis (2020) suggests that regressing forecasts on nowcasts can identify the perceived persistence of the AR(1) process. The basic idea behind this approach is to control the impact of perceived signal precision using two expectations reported contemporaneously. For instance, the nowcast and forecast of growth reported in the same period is such a pair of expectations. However, when the explicit form of the law of motion is unknown, this simple method still needs to be adapted to accommodate more general environments.

The adapted method involves examining the symmetry of the cross-covariance matrix between the expectations and their corresponding actual outcomes. The vector of expectations in the cross-covariance is still reported contemporaneously to help control the impact of perceived signal precision. In a linear Gaussian signal extraction problem, the perceived signal precision is fully captured by the perceived variance of the same signals, which is a core factor in the cross-covariance matrix and always symmetric, while the perceived and actual laws of motion only affect the remaining parts of the cross-covariance. When these two laws of motion coincide, the remaining parts of the cross-covariance are also symmetric. Therefore, by construction, the symmetry property of the perceived variance of signals carries over to the cross-covariance between expectations and outcomes.

[6] e.g. Milani (2007); Eusepi and Preston (2011).

[7] e.g. Hajdini and Kurmann (2022).

How can the method be used? In practice, the new method can be implemented with the time series of two consensus expectations along with their actual outcomes. Firstly, to reject the null hypothesis in matrix form, it is sufficient to reject an element of it, which is based on only a pair of expectations and their actual outcomes. Secondly, the new method avoids computing the covariance of two expectations, resulting in the absence of nonlinear terms about expectations in the null hypothesis. This feature allows for the linear aggregation of individual expectations to the consensus level. Consequently, this method is suitable for expectation surveys that do not cover a wide range of macroeconomic indicators and have only imprecise survey response at individual forecaster level, such as the Michigan Survey of Consumers.

By appropriately selecting consensus expectations, the method can detect various forms of misperceived laws of motion. For instance, when the two expectations are the forecasts and nowcasts of the same indicator, the method captures the discrepancies across forecast horizons. Alternatively, when the two expectations are the forecasts of two indicators, the method captures discrepancies related to variable comovement. The three applications below are designed to represent the two types of misperceived law of motion both separately and jointly. In each of the applications, the new method helps quickly narrow down the set of plausible models that account for the evidence of misperceived law of motion.

Application 1 aims to understand why consumers show little variations in their growth forecasts in the Michigan Survey of Consumers. In raw data, the consensus nowcasts are highly correlated with the actual outcomes, which indicates that consumers do observe the changes of economic conditions in the past. In the same time, the actual outcome of output growth is auto-correlated. If the consumers understand such auto-correlation, they should have their forecasts correlated with the current outcomes as well. The new method in this paper proves that it is not the case empirically both in aggregate and in all subsamples. A parsimonious model shows that such naive growth expectations can propagate an AR(1) demand shock to generate an AR(2) output dynamics.

Application 2 investigates why consumers' inflation and growth forecasts exhibits negative correlation in the Michigan Survey of Consumers, an empirical pattern not found in the corresponding actual outcomes (Candia, Coibion and Gorodnichenko, 2020). Through the lens of the method in this paper, I find that the negative correlation between inflation forecasts and growth outcomes is crucial for understanding such anomaly. A simple model shows that consumers paying no attention to supply shocks can only explain the negative correlation between the two forecasts, while consumers paying no attention to how supply shocks affect inflation can explain the negative correlation between inflation forecasts and growth outcomes additionally.

Applications 3 is designed to combine a list of statistics derived from the new method to understand the interest rate forecasts of professional forecasters. My method provides a large number of statistics across both forecast horizons and variable comovement, each of which can reject the identity between perceived and actual laws of motion. By using them as targets for structural estimation, I demonstrate that although a simple VAR(1) model is sufficient to model the actual law of motion of the 3-month treasury rate, there is a puzzling reversal of signs in the correlation between forecasts and current outcomes across forecast horizons, which goes beyond what a misperceived VAR model can explain.

In summary, consumers tend to oversimplify the process of growth and inflation forecasts, while professional forecasters tend to overcomplicate the dynamics of interest rates.

Related Literature. This paper contributes to a broader literature that investigates expectation formation processes using survey-based data of subjective expectations.

In terms of the methodology, the most relevant studies are Coibion and Gorodnichenko (2015); Bordalo et al. (2020), both of which use a simple regression to test the null hypothesis and guide the modeling of expectation formation process. Coibion and Gorodnichenko (2015) focus on information rigidity, while Bordalo et al. (2020) focus on diagnostic expectations. My paper has the same flavor of providing evidence not depending on the specific model structure but focuses on misperceived law of motion.

In terms of the research questions, the most relevant studies are Angeletos et al. (2020); Farmer et al. (2023); Xie (2023), all of which identify the misperceived law of motion in an explicitly specified model. In contrast, my paper focuses on how to provide quick evidence about misperceived law of motion, without much prior knowledge on the model structure.

In terms of the ultimate goal, my paper tries to provide the decisive evidence useful for the modeling of expectation formation in dynamic general equilibrium models. The existing studies that also use survey expectations to discipline the model include Milani (2011), Eusepi and Preston (2011), Eusepi and Preston (2018), Qiu (2019), Bhandari, Borovička and Ho (2019), Afrouzi (2020), Angeletos and Huo (2021), Chahrour and Gaballo (2021), Iovino and Sergeyev (2021), Bianchi, Ilut and Saijo (2023), L'Huillier, Singh and Yoo (2023), Huo et al. (2023) and Pei (2023).

My paper is also related to the literature that makes credible interpretations of the data without imposing specific model structures, such as McKay and Wolf (2023) and Barnichon and Mesters (2023). My paper remains silent on whether the standard errors of the tests are miscalculated due to the overlook of persistent regimes, as in Hajdini and Kurmann (2022). Nor does it touch how expectations affect decisions, as in Bachmann, Berg and Sims (2015).

The rest of this paper is organized as follows. Section 2 specifies the general framework of this paper and uses an AR(1) example to make a connection to the literature. Section 3 derives the general theoretical observation step by step and discusses extended issues. Section 4 provide three applications of the method, respectively. Section 5 concludes and lists open questions for future studies.

2 Framework

This section presents the general linear Gaussian framework that serves as the foundation for the new method proposed in this paper. It starts by introducing the primitives of this framework and subsequently employs an AR(1) law of motion to exemplify the basic idea of this method.

2.1 Primitives

The framework comprises two key components: an actual law of motion that characterizes how macroeconomic indicators evolve and a signal extraction problem that describes how forecasters form expectations.

Actual law of motion. Consider a vector of covariance stationary macroeconomic states $\mathbf{x}_t \in \mathbb{R}^n$ that evolves over time according to the following VAR(1) process.

$$\mathbf{x}_t = \boldsymbol{\Phi}\mathbf{x}_{t-1} + \boldsymbol{\Sigma}\boldsymbol{\epsilon}_t, \qquad \boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{I}), \tag{1}$$

in which $\boldsymbol{\Phi}$ is the transition matrix, $\boldsymbol{\Sigma}$ is the standard deviation, and $\boldsymbol{\epsilon}_t$ is the vector of i.i.d. aggregate shocks. The vector of macroeconomic indicators of interest $\mathbf{y}_t$ is determined by the following linear transformation

$$\mathbf{y}_t = \boldsymbol{\Psi}\mathbf{x}_t. \tag{2}$$

Denote $\Xi \equiv (\boldsymbol{\Sigma}, \boldsymbol{\Phi}, \boldsymbol{\Psi})$ as the collection of parameters that fully characterizes the actual law of motion for the process $\{\mathbf{y}_t\}$. Linearity of the framework requires Ξ to be time-invariant.

Signal extraction. Forecasters are indexed by $i \in \{1, 2, 3, \cdots, M\}$. At period $t \in \mathbb{N}$, each forecaster i collects a vector of noisy signals $\mathbf{s}_{i,\ell,t}$ for observables $\mathbf{y}_{t-\ell}$ at each lag $\ell \in \mathbb{N}$.

$$\mathbf{s}_{i,\ell,t} = \mathbf{y}_{t-\ell} + \boldsymbol{\Omega}_{i,\ell}\mathbf{e}_{i,\ell,t}, \qquad \mathbf{e}_{i,\ell,t} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}). \tag{3}$$

Here, $\mathbf{e}_{i,\ell,t}$ is a vector of idiosyncratic i.i.d. noise and $\mathbf{\Omega}_{i,\ell}$ is the inverse signal precision. The collection of signals used by forecast i to make predictions at period t is a stacked vector

$$\mathbf{w}_{i,t} \equiv \left(\mathbf{s}_{i,0,t}^{\mathsf{T}}, \mathbf{s}_{i,1,t}^{\mathsf{T}}, \mathbf{s}_{i,2,t}^{\mathsf{T}}, \cdots\right)^{\mathsf{T}}. \tag{4}$$

Let $\mathbf{\Theta}_i \equiv (\Xi, \{\mathbf{\Omega}_{i,\ell}\}_\ell)$ represent the set of actual parameter values relevant to forecaster i and $\widetilde{\mathbf{\Theta}}_i \equiv (\widetilde{\Xi}_i, \{\widetilde{\mathbf{\Omega}}_{i,\ell}\}_\ell)$ be the perceived values, with $\widetilde{\Xi}_i \equiv (\widetilde{\mathbf{\Sigma}}_i, \widetilde{\mathbf{\Phi}}_i, \widetilde{\mathbf{\Psi}}_i)$. The expectations of forecaster i on $\mathbf{y}_{t+h}$ formed at period t is the conditional expectation projection of outcomes $\mathbf{y}_{t+h}$ onto observed noisy signals $\mathbf{w}_{i,t}$, which is denoted as $\widetilde{\mathbf{y}}_{i,t+h|t}$.

$$\widetilde{\mathbf{y}}_{i,t+h|t} \equiv \widetilde{\mathbb{E}}_{i,t}[\mathbf{y}_{t+h}] = \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t}; \widetilde{\mathbf{\Theta}}_i] \quad \text{for} \quad \forall h \in \mathbb{N}, \tag{5}$$

where $\widetilde{\mathbb{E}}_{i,t}$ denotes the subjective expectation operator. Forecaster i computes the projection as if the parameters are $\widetilde{\mathbf{\Theta}}_i$ instead of $\mathbf{\Theta}_i$. Full information requires $\widetilde{\mathbf{\Omega}}_{i,\ell} = \mathbf{\Omega} = \mathbf{0}$ while RE requires $\widetilde{\mathbf{\Theta}}_i = \mathbf{\Theta}$. Under rational expectations, we denote

$$\mathbf{y}_{i,t+h|t} \equiv \mathbb{E}_{i,t}[\mathbf{y}_{t+h}] = \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t}; \mathbf{\Theta}_i] \quad \text{for} \quad \forall h \in \mathbb{N}. \tag{6}$$

The average of $\widetilde{\mathbf{y}}_{i,t+h|t}$ and $\mathbf{y}_{i,t+h|t}$ across i are denoted as $\widetilde{\mathbf{y}}_{t+h|t}$ and $\mathbf{y}_{t+h|t}$, respectively.

Definition 1 (misperceived parameters). *For forecaster $i \in \{1, 2, \cdots, M\}$, misperceived law of motion is a parameter $\widetilde{\Xi}_i$ that is different from Ξ, while misperceived signal precision is a collection of parameters $\{\widetilde{\mathbf{\Omega}}_{i,\ell}\}_\ell$ that is different from $\{\mathbf{\Omega}_{i,\ell}\}_\ell$.*

2.2 AR(1) Examples

The framework in Section 2.1 is flexible enough to encompass many well-known expectation formation models as special cases. In this subsection, I focus on a particular example with an AR(1) law of motion to establish a connection between this framework and models in the existing literature and then briefly explain the basic idea of the new method.

Example 1 (AR(1) law of motion). *$\{\mathbf{x}_t, \mathbf{y}_t, \boldsymbol{\epsilon}_t, \boldsymbol{\Sigma}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Omega}_{i,\ell}, \widetilde{\boldsymbol{\Sigma}}_i, \widetilde{\boldsymbol{\Phi}}_i, \widetilde{\boldsymbol{\Psi}}_i, \widetilde{\boldsymbol{\Omega}}_{i,\ell}\}$ are all scalars. More specially, $\boldsymbol{\Sigma} = \widetilde{\boldsymbol{\Sigma}}_i = \boldsymbol{\Psi} = \widetilde{\boldsymbol{\Psi}}_i = 1$ and $(\boldsymbol{\Omega}_{i,\ell}, \widetilde{\boldsymbol{\Omega}}_{i,\ell}, \widetilde{\boldsymbol{\Phi}}_i) = (\boldsymbol{\Omega}, \widetilde{\boldsymbol{\Omega}}_\ell, \widetilde{\boldsymbol{\Phi}})$.*

In this example, the perceived and actual persistence of the AR(1) process $(\widetilde{\boldsymbol{\Phi}}, \boldsymbol{\Phi})$ fully characterize the perceived and actual laws of motion, while the perceived inverse precision $\{\widetilde{\boldsymbol{\Omega}}_\ell\}$ captures a flexible perceived information structure. Example 1 is rich enough to nest both Angeletos et al. (2020) and Bordalo et al. (2020) as special cases.

Connection with the Literature. When $\widetilde{\boldsymbol{\Omega}}_\ell = \widetilde{\boldsymbol{\Omega}}$, Example 1 is identical to the framework of Angeletos et al. (2020), in which $\boldsymbol{\Omega} > 0$ captures noisy information, $\widetilde{\boldsymbol{\Omega}} \neq \boldsymbol{\Omega}$ represents misperceived signal precision, and $\widetilde{\boldsymbol{\Phi}} \neq \boldsymbol{\Phi}$ models misperceived law of motion. Angeletos et al. (2020) find $\widetilde{\boldsymbol{\Phi}} > \boldsymbol{\Phi} > 0$ can be detected by the over-reaction of forecasts to exogenous shocks after a few quarters. Yet, this method becomes less informative when $0 < \widetilde{\boldsymbol{\Phi}} < \boldsymbol{\Phi}$.

When $\widetilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}$, $\widetilde{\boldsymbol{\Omega}}_0^2 < \boldsymbol{\Omega}^2 \in \mathbb{R}$, and $\widetilde{\boldsymbol{\Omega}}_\ell^2 = \boldsymbol{\Omega}^2$ for $\ell \geq 1$, Example 1 is observationally equivalent to the diagnostic expectation model in Bordalo et al. (2020). The model exhibits time-inconsistent misperceived signal precision. Forecasters underestimate the variance of the currently observed signals but remain unbiased on the signals received in the past. The framework in Section 2.1 can still encompass the diagnostic expectations model as a special example because the signal extraction problem is viewed as a one-shot projection of actual outcomes onto observed noise signals instead of a Kalman filter that updates expectations recursively.

Basic idea of the new method. Regardless of the explicit structure of $\{\widetilde{\boldsymbol{\Omega}}_\ell\}$ in Example 1, the consensus forecasts and nowcasts obviously satisfy Lemma 1.

Lemma 1 (Forecast decomposition). *In the model of Section 2.1,*

$$\widetilde{\mathbf{y}}_{t+h|t} = \widetilde{\boldsymbol{\Phi}}^h \cdot \widetilde{\mathbf{y}}_{t|t}. \tag{7}$$

Proof. See Section A.1. □

Based on Lemma 1, $\widetilde{\mathbf{\Phi}}$ can be estimated by regressing the consensus forecasts $\widetilde{\mathbf{y}}_{t+1|t}$ on the corresponding nowcasts $\widetilde{\mathbf{y}}_{t|t}$ (Reis, 2020). The discrepancies between the perceived and actual laws of motion can be detected through

$$\left(\frac{\widetilde{\mathbf{\Phi}}}{\mathbf{\Phi}}\right)^h = \frac{cov(\widetilde{\mathbf{y}}_{t+h|t}, \widetilde{\mathbf{y}}_{t|t})}{var(\widetilde{\mathbf{y}}_{t|t})} \cdot \frac{var(\mathbf{y}_t)}{cov(\mathbf{y}_{t+h}, \mathbf{y}_t)}. \tag{8}$$

This method has several shortcomings. First, equation (8) is unnecessarily too complex. Second, the result is sensitive to the measurement errors of survey expectations. Third, the method is applicable only when the actual law of motion is known (see Section B.1 for an example). To address all of these shortcomings, I propose a new method in Proposition 1.

Proposition 1 (Relative perceived persistence)**.** *In the model of Section 2.1,*

$$\left(\frac{\widetilde{\mathbf{\Phi}}}{\mathbf{\Phi}}\right)^h = \frac{\widetilde{\mathbf{\Phi}}^h \cdot cov(\widetilde{\mathbf{y}}_{t|t}, \mathbf{y}_t)}{\mathbf{\Phi}^h \cdot cov(\mathbf{y}_t, \widetilde{\mathbf{y}}_{t|t})} = \frac{cov(\widetilde{\mathbf{y}}_{t+h|t}, \mathbf{y}_t)}{cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{t|t})}. \tag{9}$$

Proof. See Section A.2. □

Equation (9) essentially uses two expectations conditioned on the same information set to control the impact of perceived signal precision. In equation (9), consensus nowcast $\widetilde{\mathbf{y}}_{t|t}$ absorbs the influence of perceived inverse signal precision $\widetilde{\mathbf{\Omega}}_\ell$. The covariance $cov(\widetilde{\mathbf{y}}_{t|t}, \mathbf{y}_t)$ in both the numerator and the denominator can eliminate the influence of $\widetilde{\mathbf{\Omega}}_\ell$ completely, leaving the comparison between the perceived law of motion and the actual law of motion isolated from the misperceived signal precision.

Equation (9) addresses all of the three shortcomings of equation (8). First, it simplifies the formula by eliminating the repeated use of $\widetilde{\mathbf{y}}_{t|t}$ and $\mathbf{y}_t$. Second, equation (9) is immune to the measurement errors of survey expectations because it avoids computing the covariance of two expectations. As a result, the measure errors wash out in the covariance. Third, the general version of equation (9) is applicable to more complex but unknown actual law of motion, which is the primary focus of the discussion in Section 3.

3 Methodology

This section generalizes the method in Proposition 1 to accommodate a more general but unspecified actual law of motion. It demonstrates the essence of the method, explains how to implement it in practice, and discusses several extensions.

3.1 Method

This subsection demonstrates the essence of the method proposed in this paper by building up the theoretical findings step-by-step. It starts with the well-known approach of forecast error predictability designed to test rational expectations. It proceeds by modifying the approach to control the role of perceived signal precision and to isolate the comparison of the perceived and actual laws of motion. This subsection also constructs a pair of covariance statistics to make the comparison observable in the data.

Forecast error predictability. I develop the new method from the well-known approach of forecast error predictability for individual-level data (e.g. Bordalo et al., 2020). Consider the projections of actual outcomes onto observed noisy signals for both subjective expectations $\widetilde{\mathbf{y}}_{i,t+h|t}$ and rational expectations $\mathbf{y}_{i,t+h|t}$. According to the model in Section 2.1,

$$\widetilde{\mathbf{y}}_{i,t+h|t} = \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t};\widetilde{\boldsymbol{\Theta}}_i] = \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t};(\widetilde{\Xi}_i,\{\widetilde{\boldsymbol{\Omega}}_{i,\ell}\}_\ell)], \tag{10}$$

$$\mathbf{y}_{i,t+h|t} = \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t};\boldsymbol{\Theta}_i] = \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t};(\Xi,\{\boldsymbol{\Omega}_{i,\ell}\}_\ell)]. \tag{11}$$

The forecast error predictability regressions using individual-level data (e.g. Bordalo et al., 2020) estimate $\mathbb{E}[\mathbf{y}_{t+h} - \widetilde{\mathbf{y}}_{i,t+h|t}|\mathbf{w}_{i,t};\boldsymbol{\Theta}_i]$. Under rational expectations, $\widetilde{\boldsymbol{\Theta}}_i = \boldsymbol{\Theta}_i$, and

$$\begin{aligned}\mathbb{E}[\mathbf{y}_{t+h} - \widetilde{\mathbf{y}}_{i,t+h|t}|\mathbf{w}_{i,t};\boldsymbol{\Theta}_i] &= \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t};\boldsymbol{\Theta}_i] - \mathbb{E}[\widetilde{\mathbf{y}}_{i,t+h|t}|\mathbf{w}_{i,t};\widetilde{\boldsymbol{\Theta}}_i] \\ &= \mathbf{y}_{i,t+h|t} - \widetilde{\mathbf{y}}_{i,t+h|t} = \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t};\boldsymbol{\Theta}_i] - \mathbb{E}[\mathbf{y}_{t+h}|\mathbf{w}_{i,t};\widetilde{\boldsymbol{\Theta}}_i] = 0.\end{aligned} \tag{12}$$

$\widetilde{\boldsymbol{\Theta}}_i = \boldsymbol{\Theta}_i$ requires both $\widetilde{\Xi}_i = \Xi$ for law of motion and $\widetilde{\boldsymbol{\Omega}}_{i,\ell} = \boldsymbol{\Omega}_{i,\ell}$ for signal precision.

Controlling unknown perceived signal precision. To separately detect $\widetilde{\Xi}_i \neq \Xi$, we need to control the role of $\widetilde{\mathbf{\Omega}}_{i,\ell}$, which requires a linear Gaussian structure and the signal extraction problem in the model of Section 2.1. For exposition purpose, I consider a case in which the stacked noisy signals $\mathbf{w}_{i,t}$ only has finite memory. This case can be viewed as the truncation approximation to the original signal extraction problem in Section 2.1, which allows us to illustrate the essence of the method more transparently.

Under the linear Gaussian signal extraction problem, equations (10) and (11) becomes

$$\widetilde{\mathbf{y}}_{i,t+h|t} = cov(\mathbf{y}_{t+h}, \mathbf{w}_{i,t}|\widetilde{\Xi}_i) \cdot var(\mathbf{w}_{i,t}|\widetilde{\mathbf{\Theta}}_i)^{-1} \cdot \mathbf{w}_{i,t}, \tag{13}$$

$$\mathbf{y}_{i,t+h|t} = cov(\mathbf{y}_{t+h}, \mathbf{w}_{i,t}|\Xi) \cdot var(\mathbf{w}_{i,t}|\mathbf{\Theta}_i)^{-1} \cdot \mathbf{w}_{i,t}. \tag{14}$$

Note that the perceived signal precision only affects $var(\mathbf{w}_{i,t}|\widetilde{\mathbf{\Theta}}_i)$ but not $cov(\mathbf{y}_{t+h}, \mathbf{w}_{i,t}|\widetilde{\Xi}_i)$, because the idiosyncratic noise $\mathbf{e}_{i,\ell,t}$ in signal $\mathbf{w}_{i,t}$ is not correlated with the actual outcomes $\mathbf{y}_t$. To control the perceived signal precision in $\widetilde{\mathbf{\Theta}}_i$ and isolate the comparison between the perceived law of motion $\widetilde{\Xi}_i$ and the actual law of motion Ξ, we shall not compare forecasts $\widetilde{\mathbf{y}}_{i,t+h|t}$ and $\mathbf{y}_{i,t+h|t}$ directly, but instead compare the following two expressions

$$cov(\mathbf{y}_{t+h}, \mathbf{w}_{i,t}|\widetilde{\Xi}_i) \cdot var(\mathbf{w}_{i,t}|\widetilde{\mathbf{\Theta}}_i)^{-1} \cdot \mathbf{w}_{i,t}, \tag{15}$$

$$cov(\mathbf{y}_{t+h}, \mathbf{w}_{i,t}|\Xi) \cdot var(\mathbf{w}_{i,t}|\widetilde{\mathbf{\Theta}}_i)^{-1} \cdot \mathbf{w}_{i,t}. \tag{16}$$

Notably, I have replaced the actual parameters $\mathbf{\Theta}_i$ in equation (14) with $\widetilde{\mathbf{\Theta}}_i$ in equation (16), so that the differences between equations (15) and (16) must be associated with the laws of motion $\widetilde{\Xi}_i$ and Ξ_i. Still, equation (16) is not observable in the data.

Observable covariance statistics. To make the comparison between equations (15) and (16) observable in the data, I construct the covariance of these two expressions with the actual outcomes $\mathbf{y}_{t+h}$, in which the covariance is computed based on the actual law of motion Ξ and the perceived law of motion $\widetilde{\Xi}_i$, respectively.

Now, the two expressions to compare become the two covariance statistics below.

$$cov(\mathbf{y}_{t+h}, \mathbf{w}_{i,t}|\widetilde{\Xi}_i) \cdot var(\mathbf{w}_{i,t}|\widetilde{\Theta}_i)^{-1} \cdot cov(\mathbf{w}_{i,t}, \mathbf{y}_{t+h}|\Xi) = cov(\widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h}), \tag{17}$$

$$cov(\mathbf{y}_{t+h}, \mathbf{w}_{i,t}|\Xi) \cdot var(\mathbf{w}_{i,t}|\widetilde{\Theta}_i)^{-1} \cdot cov(\mathbf{w}_{i,t}, \mathbf{y}_{t+h}|\widetilde{\Xi}_i) = cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{i,t+h|t}). \tag{18}$$

Equations (17) and (18) indicate that when the perceived law of motion is identical to the actual law of motion, i.e., $\widetilde{\Xi}_i = \Xi$, we have $cov(\widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h}) = cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{i,t+h|t})$. In other words, the symmetry of $cov(\widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h})$ is the null hypothesis of $\widetilde{\Xi}_i = \Xi$.

This null hypothesis can be tested using the time series of two consensus expectations along with their actual outcomes. Individual-level expectations are not needed because of the following aggregation.

$$\begin{aligned} M \cdot cov(\widetilde{\mathbf{y}}_{t+h|t}, \mathbf{y}_{t+h}) &= cov(\textstyle\sum_{i=1}^{M} \widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h}) = \textstyle\sum_{i=1}^{M} cov(\widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h}) \\ &= \textstyle\sum_{i=1}^{M} cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{i,t+h|t}) = cov(\mathbf{y}_{t+h}, \textstyle\sum_{i=1}^{M} \widetilde{\mathbf{y}}_{i,t+h|t}) = M \cdot cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{t+h|t}). \end{aligned} \tag{19}$$

This aggregation is allowed because the design of the covariance statistics $cov(\widetilde{\mathbf{y}}_{t+h|t}, \mathbf{y}_{t+h})$ avoids computing the covariance of two expectations, and therefore has no quadratic terms about expectations. Additionally, to reject a matrix equation, we only need to reject an element of it. As a result, only a pair of objects in $\mathbf{y}_{t+h}$ is needed when testing $cov(\widetilde{\mathbf{y}}_{t+h|t}, \mathbf{y}_{t+h}) = cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{t+h|t})$.

Example 2 (Minimal data requirements)**.** *Let* $\mathbf{y}_{t+1} = (\ln \frac{GDP_{t+1}}{GDP_t}, \ln \frac{GDP_t}{GDP_{t-1}}, \cdots)^\intercal$, *then misperceived law of motion can be detected if*

$$cov\left(\frac{1}{M}\sum_{i=1}^{M} \widetilde{\mathbb{E}}_{i,t} \ln \frac{GDP_{t+1}}{GDP_t}, \ln \frac{GDP_t}{GDP_{t-1}}\right) \neq cov\left(\ln \frac{GDP_{t+1}}{GDP_t}, \frac{1}{M}\sum_{i=1}^{M} \widetilde{\mathbb{E}}_{i,t} \ln \frac{GDP_t}{GDP_{t-1}}\right). \tag{20}$$

With all of the preparation above, we are now ready to summarize the main theoretical findings of this paper in Theorem 1.

Theorem 1 (Detecting misperceived law of motion). *In any models consistent with the framework of Section 2.1, $\widetilde{\Xi}_i \neq \Xi$ for some $i \in \{1, 2, \cdots, M\}$ is detected if for some j, k, and $h \in \mathbb{N}_{>0}$,*

$$cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_{t+h}(k)) \neq cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t+h|t}(k)), \tag{21}$$

in which $\mathbf{y}_{t+h}(j)$ denotes the jth element of the outcome vector $\mathbf{y}$ at period $t + h \geq t$.

Proof. See Section A.3. □

Theorem 1 is a powerful method to detect misperceived law of motion because it does not require prior knowledge of $(\mathbf{\Omega}_{i,\ell}, \widetilde{\mathbf{\Omega}}_{i,\ell})$ or Ξ within the general framework of Section 2.1. This nice property enables the new method to provide stylized facts to guide the modeling of expectation formation in full-fledged macroeconomic models. It also rules out all other possibilities[8] so that the stylized facts are decisive evidence of misperceived law of motion.

3.2 Practical issues

Theorem 1 provides a test for the identity between the perceived and actual laws of motion. How can we tell whether a rejection of the null hypothesis is statistically significant or not? What can we do after rejecting the null hypothesis? This subsection addresses these issues.

Statistical inference. In practice, Theorem 1 can be implemented with auxiliary moment condition $\mathbb{E}[m_t] \neq 0$, in which $t \in \{1, 2, \cdots, T\}$ is the sample range and m_t is given by

$$m_t = \left(\widetilde{\mathbf{y}}_{t+h|t}(j) - \frac{1}{T}\sum_{t=1}^{T}\widetilde{\mathbf{y}}_{t+h|t}(j)\right)\left(\mathbf{y}_{t+h}(k) - \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_{t+h}(k)\right) \\ - \left(\mathbf{y}_{t+h}(j) - \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_{t+h}(j)\right)\left(\widetilde{\mathbf{y}}_{t+h|t}(k) - \frac{1}{T}\sum_{t=1}^{T}\widetilde{\mathbf{y}}_{t+h|t}(k)\right). \tag{22}$$

[8]In contrast, to explain two seemly conflicting coefficients in forecast error predictability regressions, Kohlhas and Walther (2021) extend the actual law of motion beyond AR(1), while Angeletos and Huo (2021) introduce misperceived law of motion on an AR(1) process. These approaches are not designed to rule out other possibilities.

Corollary 1 (Simple Test). *In any models consistent with the framework of Section 2.1, $\widetilde{\Xi}_i \neq \Xi$ for some $i \in \{1, 2, \cdots, M\}$ is detected if $Pr\left(\lim_{T\to+\infty} \frac{1}{T}\sum_{t=1}^{T} m_t = 0\right) \neq 1$.*

Proof. See Section A.5. □

Corollary 1 makes the test in Theorem 1 straightforward to implement. First, construct the auxiliary time series $\{m_t\}$ from the actual outcomes and the consensus expectations of two indicators $\mathbf{y}_{t+h}(j)$ and $\mathbf{y}_{t+h}(k)$ in the law of motion. Second, estimate the mean of $\{m_t\}$ in regression taking into account the autocorrelation of the sample $\{m_t\}_{t=1}^{T}$. The standard errors of the regression coefficients can be used for statistical inference.

Informative statistics. After detecting the presence of misperceived law of motion using Theorem 1, we can further construct two types of informative statistics based on Theorem 1 below to measure the discrepancies between the perceived and actual laws of motion.

$$z_h^{term}(j,k) \equiv \frac{cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_t(k))}{cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k))}, \tag{23}$$

$$z_h^{como}(j,k) \equiv \frac{cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_{t+h}(k)) - cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t+h|t}(k))}{std(\mathbf{y}_{t+h}(j)) \cdot std(\mathbf{y}_{t+h}(k))}. \tag{24}$$

Corollary 2 (Informative statistics). *In any models consistent with the framework of Section 2.1, $\widetilde{\Xi}_i \neq \Xi$ for some $i \in \{1, 2, \cdots, M\}$ is detected if $z_h^{term}(j,k) \neq 1$ or $z_h^{como}(j,k) \neq 0$.*

Proof. See Section A.6. □

Corollary 2 implies that $z_h^{term}(j,k)$ and $z_h^{como}(j,k)$ are normalized discrepancies between the perceived and actual laws of motion.

Although $z_h^{term}(j,k)$ does not have analytical solution when $\widetilde{\Xi}_i = \Xi$ is violated, it is still informative about misperceived law of motion across forecast horizons. In the special case of AR(1) law of motion, according to Proposition 1, $z_h^{term}(1,1) = (\widetilde{\mathbf{\Phi}}/\mathbf{\Phi})^h$ can be interpreted as the measure of accumulated misperceptions in AR(1) persistence across forecast horizon h. With a general law of motion, $z_h^{term}(j,k)$ still captures the accumulation of misperceptions in terms of the covariance of expectations and actual outcomes across forecast horizons.

$z_h^{como}(j,k)$ is different from $z_h^{term}(j,k)$ in two aspects. First, $z_h^{como}(j,k)$ is the normalized difference of two covariance statistics, so that it applies to the situation when one of the two covariance statistics is zero. Second, $z_h^{como}(j,k)$ focuses on the misperceived law of motion about contemporaneous variable comovement.

$z_h^{term}(j,k)$ is designed for the survey data that has expectations over multiple horizons, such as the survey of professional forecasters, while $z_h^{como}(j,k)$ is designed for the survey data that has expectations on multiple variables at the same horizon, such as the Michigan Survey of Consumers.

3.3 Discussion

What is the boundary within which the new method is applicable? Theoretically, the linear Gaussian framework in Section 2.1 is the only restriction. In practice, there are still several issues to clarify for extensions.

Beyond the framework. What if the general framework in Section 2.1 does not hold? In such a situation, the method in Theorem 1 can still test rational expectations. However, it is no longer clear whether the misperceptions originate from the law of motion or the signal precision. More precisely, we have Theorem 2.

Theorem 2 (Rational expectations test)**.** *Violation of rational expectations is detected if for some j, k, and $h \in \mathbb{N}_{>0}$, $cov(\tilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_{t+h}(k)) \neq cov(\mathbf{y}_{t+h}(j), \tilde{\mathbf{y}}_{t+h|t}(k))$.*

Proof. See Section A.4. □

Public signal. In the model of Section 2.1, $\mathbf{e}_{i,\ell,t}$ is an i.i.d. noise and $\mathbf{s}_{i,\ell,t}$ is a private signal. A public signal can break the i.i.d. assumption on the noise in two aspects (e.g. Angeletos and La'O, 2013).

First, public signal noise may be identical across all forecasters. This assumption does not invalidate Theorem 1 because the method remains valid at the individual level. Therefore, whether the signal noise cancels out in aggregate is inconsequential as long as it is not correlated with the actual outcomes $\mathbf{y}_t$.

Second, the public signal noise may be correlated with the aggregate outcomes $\mathbf{y}_t$. This correlation occurs when forecasters collectively respond to the public signal noise, leading to changes in $\mathbf{y}_t$. In this scenario, Theorem 1 no longer holds due to the dependence of the covariance between the outcome $\mathbf{y}_t$ and the signal $\mathbf{w}_{i,t}$ on the inverse signal precision $\mathbf{\Omega}_{i,\ell}$ in equations (13) and (14).

This issue can be circumvented by incorporating the public signal noise into aggregate states $\mathbf{x}_t$. As a result, the covariance between outcome $\mathbf{y}_t$ and signal $\mathbf{w}_{i,t}$ no longer depends on the inverse signal precision $\mathbf{\Omega}_{i,\ell}$. The new method can still detect misperceived law of motion by interpreting public signal noise as a component of the law of motion.

Learning. One type of learning model focuses on learning about the actual law of motion Ξ using precise observations, rather than learning about the states $\mathbf{x}_t$ through noisy signals (e.g. Milani, 2007; Eusepi and Preston, 2011, 2018; Farmer et al., 2023). Is the new method still applicable to such a framework?

In principle, the answer is NO because the presence of time-varying actual or perceived law of motion would violate the framework presented in Section 2.1. Nonetheless, the new method remains capable of testing rational expectations as outlined in Theorem 2. In practice, when the learning process is nearly stationary, we may still view the linear Gaussian framework in Section 2.1 as a linear approximation of the learning models. Consequently, the new method based on Theorem 1 remains valid up to first order approximation.

Bias of Mean. Some expectation formation models also emphasize systematic bias about the mean (Bhandari et al., 2019; Huo et al., 2023). When the bias is constant over time, the new method in this paper completely overlooks it by focusing on only covariance statistics.

4 Applications

This section designs three applications to highlight two types of misperceived law of motion both separately and jointly, to cover two well-known expectation surveys, and to shed light on three empirical anomalies discussed in the literature.

4.1 Naive Growth Forecasts

In the Michigan Survey of Consumers, the 1-year forward forecasts on business conditions changes exhibit little variation. Why do consumers have such naive expectations? What is the macroeconomic implication of such expectations? This subsection applies the statistic $z_h^{term}(j,k)$ defined in Section 3.2 to address the two questions. This application provides the simplest example of misperceived law of motion across forecast horizons.

Data. The Michigan Survey of Consumers draws the sample nationally each month based on random digit dialing of cellular telephone numbers. It asks for categorical assessments of several macroeconomic conditions from the respondents. The answer for business conditions change is restricted to "better", "worse", "the same", and "do not know".

In this application, I use the business condition changes as a proxy variable for the real GDP growth rate (outcomes). The Michigan Survey of Consumers provides corresponding variables, namely the recalled business condition changes from a year ago (nowcasts) and the expected 1-year forward business condition changes (forecasts). To convert these nowcasts and forecasts from categorical to numerical values, I apply the procedure outlined in Mankiw, Reis and Wolfers (2003) (See Section C for the details).

This application utilizes the nowcasts and forecasts transformed from the corresponding categorical data in the Michigan Survey of Consumers. The left panel of Figure 1 compares the nowcasts of business condition change to the actual outcomes of real GDP growth rate. Remarkably, the two corresponding lines in the figure exhibit a nearly perfect overlap during the 1985-2011 period, indicating that the business condition change is a good proxy variable for the real GDP growth rate. To mitigate potential drift of long-term growth rate, the focus is placed on the pre-2012 sample. Additionally, the most recent release of the real GDP growth rate rather than the real-time growth rate is used as the outcomes to maximize the correlation between the nowcasts and the outcomes.

[Figure 1 about here.]

Despite the close match between outcomes and nowcasts in consensus level, there is a significant nowcast dispersion in the left panel of Figure 1. This observation indicates the presence of dispersed information rather than sticky information. In other words, the data feature suggests that while the information available may be noisy in reality, forecasters do not adequately discount their private signals during the process of signal extraction.

Despite the large variations in the consensus nowcasts, the forecasts display little variations both in consensus level and in dispersion. This empirical pattern suggests that while consumers appear to be aware of past observations at the consensus level, they do not effectively utilize this information to forecast the future. Nevertheless, it is important to note that the raw data presented in Figure 1 alone is not sufficient to draw conclusive evidence regarding consumers' failure to utilize observed information in forecasting.

Test. To provide decisive evidence, I denote $g_t \equiv 100 \cdot \ln \frac{GDP_t}{GDP_{t-4}}$ as the macroeconomic indicator to focus on and consider the following three statistics

$$\left(z^{auto}, z^{now}, z^{fore}\right) \equiv \left(cov(g_{t+4}, g_t), cov(g_{t+4}, \widetilde{g}_{t|t}), cov(\widetilde{g}_{t+4|t}, g_t)\right), \tag{25}$$

in which $\widetilde{g}_{t|t}$ is the transformed nowcast and $\widetilde{g}_{t+4|t}$ is the transformed forecast. The Michigan Survey of Consumers provides cross-sectional samples for subsample analysis. Table 1 summarizes statistics built on equation (25). It reports the results for both the full sample and the subsamples grouped by age, region, gender, income, and education.

[Table 1 about here.]

The full sample result is evident. First, z^{auto} is significantly above zero, which indicates that the annual growth rate of real GDP does have clear pattern of autocorrelation. Second, there is no clear evidence of information rigidities as $z^{auto} - z^{now}$ is not significantly above zero. Third, there is evidence of misperceived law of motion as $z^{now} - z^{fore}$ is positive with relatively small standard errors. Moreover, $z^{now}/z^{auto} \geq 1$ confirms the pattern of Figure 1 that consumers do not interpret the signals they receive via signal extraction. $z^{fore}/z^{now} \approx$

0 implies that consumers fail to utilize the autocorrelation of GDP growth rate they observe in forecasting. Note that z^{fore}/z^{now} is a simple form of $z_h^{term}(j,k)$ with $j = k = h = 1$ in this application, and $z^{fore}/z^{now} << 1$ reveals large discrepancies between perceived and actual laws of motion across forecast horizons.

All of the findings in the full sample can also be found in each subsample, which means that the misperceived law of motion is a type of common belief distortions not sensitive to demographic characteristics. In particular, there is no evidence that the senior consumers at work, those living in the northeast, the male consumers, and those with higher income or more education understand the business conditions better. As a conclusion, it is innocuous to assume that all consumers overlook the dynamics of business condition changes in their forecasts when modeling their expectation formation process in business cycle analysis. I will use a model to demonstrate the macroeconomic implications of the such expectations.

Model. Let $y_t \equiv 100 \cdot \ln \frac{GDP_t}{GDP_{ss}}$ denote the log deviation of real GDP from the steady state. Then, we have $g_t = y_t - y_{t-4}$. The simplest way to model $z_1^{term}(1,1) \approx 0$ is to assume that consumers always perceive the future as identical to the past, i.e.,

$$\widetilde{y}_{i,t+h|t} = y_t. \tag{26}$$

To explore the implication of this forecast rules for business cycle dynamics, consider a parsimonious "Intertemporal Keynesian Cross" model following the idea of Auclert, Rognlie and Straub (2023). The consumers' optimization yields the following consumption rule (See Section B.2 for the micro-foundation).

$$c_t\left(\{\widetilde{y}_{t+h|t-1}\}_h, \{\mathbb{E}_t\eta_{t+h}\}_h\right) = \sum_{h=0}^{+\infty} \gamma^h (mpc \cdot \widetilde{y}_{t+h|t-1} + EIS \cdot \gamma \mathbb{E}_t \eta^d_{t+h}), \tag{27}$$

in which mpc is the marginal propensity to consume, EIS is the intertemporal elasticity of substitution, γ is the effective discount factor, and η^d_t is the exogenous demand wedge with an AR(1) persistence ρ.

Prices, wages, and interest rates are all constant. Firms produce whatever is demanded and transfer all revenues to the households. Market clearing in goods market requires

$$c_t\left(\{\widetilde{y}_{t+h|t-1}\}_h, \{\mathbb{E}_t\eta_{t+h}\}_h\right) = y_t. \tag{28}$$

Imposing $\sum_{h=0}^{+\infty}\gamma^h mpc < 1$ ensures the stationarity of the equilibrium.

Proposition 2. *In the model of Section 4.1, the equilibrium $\{y_t\}$ is an AR(2) process.*

Proof. See Section A.7. □

Proposition 2 indicates that the perceived law of motion in equation (26) supported by the statistics $z_1^{term}(1,1) \approx 0$ propagates demand shocks in business cycles. The mechanism is simple. In each period, the optimal consumption is driven by both the expected life-time income and the demand wedge. The rise of real GDP in last period raises the forecasts of life-time income due to the misperceived law of motion in equation (26), leading to higher consumption level in the current period. This mechanism can be summarized by

$$y_t = \underbrace{\frac{mpc}{1-\gamma}y_{t-1}}_{\text{forecasts}} + \underbrace{\gamma\cdot\frac{EIS}{1-\rho\gamma}\eta_t^d}_{\text{demand wedge}}, \tag{29}$$

which is one of the key mechanisms in Qiu (2019).

In contrast, if we do not have the empirical results $z_1^{term}(1,1) \approx 0$ support the validity of equation (26), the naive growth forecasts depicted in the right panel of Figure 1 may also be interpreted as consumers not paying any attention to macroeconomic conditions, or simply $\widetilde{y}_{i,t+h|t} = 0$ like the naive agents in Farhi and Werning (2019). In this case, the equilibrium real GDP dynamics would follow an AR(1) process instead of an AR(2) process.

This alternative result emphasizes the importance of employing the method proposed in this paper to detect a misperceived law of motion. Misinterpreting survey expectations can have a profound impact, giving rise to drastically different dynamics in business cycles.

4.2 Misaligned Inflation Expectations

In the Michigan Survey of Consumers, consumers hold the belief that inflation is linked to deteriorating economic conditions. However, this empirical pattern is not found in actual outcomes. What model element is needed to explain this empirical anomaly? This subsection utilizes the statistic $z_h^{como}(j,k)$ in Section 3.2 to address this question. This application serves as a simple example of misperceived law of motion on variable comovement.

Data. The Michigan Survey of Consumers provides data on 1-year ahead inflation expectations. However, the forecasts appear to be disconnected from the outcomes of CPI inflation, as illustrated in Figure 2. Additionally, there is a negative correlation between 1-year ahead inflation expectations and 1-year ahead real GDP growth expectations, as shown in Figure 3. This negative correlation has been documented in the literature (e.g. Bhandari et al., 2019; Candia et al., 2020). Zhao (2022) specifically highlights this correlation as the indication of misaligned inflation expectations, as it does not align with the absence of negative correlation in professional forecasts. What explains this misaligned inflation expectations? The statistics $z_h^{como}(j,k)$ can shed light on the answer.

[Figure 2 about here.]

[Figure 3 about here.]

Test. Let π_{t+4} and g_{t+4} denote the 1-year forward inflation rate and real GDP growth rate. Table 2 reports the following three statistics

$$\left(\frac{cov(\pi_{t+4}, g_{t+4})}{std(\pi_{t+4})std(g_{t+4})}, \quad \frac{cov(\widetilde{\pi}_{t+4|t}, \widetilde{g}_{t+4|t})}{std(\pi_{t+4})std(g_{t+4})}, \quad \frac{cov(\widetilde{\pi}_{t+4|t}, g_{t+4}) - cov(\pi_{t+4}, \widetilde{g}_{t+4|t})}{std(\pi_{t+4})std(g_{t+4})} \right). \quad (30)$$

The first two statistics validate the findings documented in the literature. The actual outcomes show a weakly positive correlation between inflation rate and real GDP growth rate. Conversely, consumers' 1-year ahead forecasts exhibit a negative correlation of these two variables. The third statistic is $z_4^{como}(j,k)$, which has a significant negative value in data. It suggests that incorporating a misperceived law of motion in modeling the expectation formation process is crucial to account for the empirical patterns. I will construct a simplified model in line with Zhao (2022) but with extensions to show how misperceived law of motion helps explain the misaligned inflation expectations as well as the negative $z_4^{como}(j,k)$.

[Table 2 about here.]

Model. There is a persistent demand wedge η_t^d and supply wedge η_t^s satisfying

$$\eta_t^d = \rho \cdot \eta_{t-1}^d + \epsilon_t^d, \quad \eta_t^s = \rho \cdot \eta_{t-1}^s + \sigma_s \cdot \epsilon_t^s, \quad \epsilon_t^d, \epsilon_t^s \sim i.i.d.\ \mathcal{N}(0,1). \tag{31}$$

in which $\rho \in (0,1)$ is the common persistence of the two wedges and $\sigma_s = 1$ is the standard deviation of supply shocks. The real GDP y_t and inflation π_t are determined by these two wedges in the following way

$$y_t = \eta_t^d + \eta_t^s, \tag{32}$$

$$\pi_t = \eta_t^d - \kappa \cdot \eta_t^s. \tag{33}$$

The forecasters cannot observe (y_t, π_t). They perceive η_t^s as if the parameter value of σ_s is $\widetilde{\sigma}_s \in \{0,1\}$. They also perceive $\kappa = 1$ as $\widetilde{\kappa} \in \{0,1\}$. Perceived information friction requires $\widetilde{\sigma}_s = 0$, while misperceived law of motion requires $\widetilde{\kappa} = 0$. Denote $g_{t+4} \equiv y_{t+4} - y_t$.

Proposition 3. *The model of Section 4.2 satisfies*

$$\frac{cov(\pi_{t+4}, g_{t+4})}{std(\pi_{t+4})std(g_{t+4})} = 0, \tag{34}$$

$$\frac{cov(\pi_{t+4}, \widetilde{g}_{t+4|t})}{std(\pi_{t+4})std(g_{t+4})} = -\frac{\sqrt{2}}{4}\rho^4\sqrt{1-\rho^4} \cdot (1-\widetilde{\sigma}_s), \tag{35}$$

$$\frac{cov(\widetilde{\pi}_{t+4|t}, g_{t+4})}{std(\pi_{t+4})std(g_{t+4})} = -\frac{\sqrt{2}}{4}\rho^4\sqrt{1-\rho^4} \cdot (1-\widetilde{\kappa} \cdot \widetilde{\sigma}_s), \tag{36}$$

$$\frac{cov(\widetilde{\pi}_{t+4|t}, \widetilde{g}_{t+4|t})}{std(\pi_{t+4})std(g_{t+4})} = -\frac{\sqrt{2}}{4}\rho^4\sqrt{1-\rho^4} \cdot (1-\widetilde{\kappa} \cdot \widetilde{\sigma}_s^2). \tag{37}$$

Proof. See Section A.8. □

Proposition 3 summarizes all statistics we need to understand the misaligned inflation expectations. First, the demand and supply shocks are designed to be equally important in the determinants of inflation but drive it to opposite directions. As a result, the correlation of inflation rate and real GDP growth rate is zero in realized outcomes.

Second, the correlation of the 1-year forward inflation and growth forecast is also zero under full information rational expectations, in which $\widetilde{\sigma}_s = \sigma_s = \widetilde{\kappa} = \kappa = 1$. Otherwise, each of $\widetilde{\sigma}_s = 0$ and $\widetilde{\kappa} = 0$ can make the correlation negative as in the data.

Third, without misperceived law of motion, we have $\widetilde{\kappa} = 1$ and then

$$z_4^{como}(j,k) = \frac{cov(\widetilde{\pi}_{t+4|t}, g_{t+4})}{std(\pi_{t+4})std(g_{t+4})} - \frac{cov(\pi_{t+4}, \widetilde{g}_{t+4|t})}{std(\pi_{t+4})std(g_{t+4})} = 0, \tag{38}$$

despite the misaligned inflation expectations of the consumers documented in Zhao (2022). In contrast, under misperceived law of motion in the form of $\widetilde{\kappa} = 1$,

$$z_4^{como}(j,k) = -\frac{\sqrt{2}}{4}\rho^4\sqrt{1-\rho^4} \cdot \widetilde{\sigma}_s \leq 0, \tag{39}$$

which is consistent with the last column of Table 2.

In summary, although both perceived information friction ($\widetilde{\sigma}_s = 0$) and misperceived law of motion ($\widetilde{\kappa} = 0$) have the potential to explain the misaligned inflation expectations, only misperceived law of motion ($\widetilde{\kappa} = 0$) can rationalize the comovement of expectations and realized outcomes captured by $z_4^{como}(j,k)$.

Note that the model in Section 4.2 has a different assumption on growth forecasts from that in Section 4.1. These two assumptions just capture two aspects of the growth forecasts. One captures the lack of variation, while the other one captures the misaligned correlation displayed in the small variation. Both assumptions are designed to highlight the takeaways from the corresponding applications.

4.3 Reversal of Interest Rate Forecasts

In the Survey of Professional Forecasters, the forecasted interest rate dynamics exhibit puzzling patterns. At short forecast horizons, the forecasts move in the same direction as the outcomes, but at medium horizons, they move in opposite directions. This subsection uses the statistic $z_h^{term}(j,k)$ defined in Section 3.2 to demonstrate that these puzzling patterns are indeed caused by a misperceived law of motion. Moreover, the misperception reflects the forecasters over-complicating the law of motion of interest rate dynamics. This application also illustrates that rich data can provide a multitude of statistics, each of which rejects the assumption that the perceived and actual laws of motion are identical. Furthermore, these statistics aid in identifying the perceived law of motion in structural estimation.

Data. The Survey of Professional Forecasters surveys about 40 anonymous professional forecasters each quarter. It starts from 1968, is currently run by the Federal Reserve Bank of Philadelphia, and covers the reported subjective expectations about many macroeconomic indicators such as real GDP, GDP deflator, CPI, real consumption, unemployment rate, and 3-month treasury rate.

This section focuses on two of the indicators, the unemployment rate ($UNEMP$) and the 3-month treasury rate ($tb3m$) for a couple of reasons. First, due to the monetary policy response, unemployment rate can predict a large fraction of the variations in the 3-month treasury rate. Second, the first order differences of both indicators exhibit "under-reaction" to forecast revisions at both consensus and individual levels (Bordalo et al., 2020), which makes the approach in Angeletos et al. (2020) less informative. Third, their actual outcomes have easy-to-understand law of motion well captured by a VAR(1) model.

The survey counterpart of these indicators covers $\{\widetilde{\mathbf{y}}_{i,t|t}, \widetilde{\mathbf{y}}_{i,t+1|t}, \widetilde{\mathbf{y}}_{i,t+2|t}, \widetilde{\mathbf{y}}_{i,t+3|t}, \widetilde{\mathbf{y}}_{i,t+4|t}\}$ for $\mathbf{y} = (UNEMP, tb3m)^{\intercal}$. I drop the sample before 1985 to avoid the impacts of long-term inflation drifts on the interest rate dynamics. The remaining sample exhibits clear patterns of persistent forecast errors.

Figure 4 compares the consensus expectations $\{\widetilde{\mathbf{y}}_{t+h|t}\}$ for $h \in \{-1, 0, 1, \cdots, 4\}$ with its outcomes $\{\mathbf{y}_t\}$ for $\mathbf{y} = (UNEMP, tb3m)^\intercal$. Despite the apparently persistent rise and fall of UNEMP and tb3m, the forecasters tend to overestimate UNEMP and tb3m in the rising phase and underestimate them in the falling one. The over- and underestimation increase over the forecast horizon h. Moreover, when $h = 3$ or 4, the correlation of $\widetilde{\mathbf{y}}_{t+h|t} - \widetilde{\mathbf{y}}_{t+h-1|t}$ and $\mathbf{y}_{t+h} - \mathbf{y}_{t+h-1}$ reverts to negative from positive, which is a puzzling pattern also shown in Farmer et al. (2023).

[Figure 4 about here.]

Test. Is misperceived law of motion a necessary model element to rationalize the puzzling patterns of correlation reversal? The answer is YES if we are within the general framework of Section 2.1. The method based on Theorem 1 rejects the null hypothesis $cov(y_{t+h}, \widetilde{y}_{t|t}) - cov(\widetilde{y}_{t+h|t}, y_t) = 0$ for $y \in \{\Delta UNEMP, \Delta tb3m\}$. See Table 3 for the test statistics.

[Table 3 about here.]

Does the method provide any informative guidance on how to model the expectation formation process? The answer is also YES since the statistics $z_h^{term}(j,k)$ can be used to tell what aspects of the model still needs improvement.

Model. I consider three models with both information frictions and potentially misperceived parameters. Model 1 assumes $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}$, Model 2 removes this restriction, and Model 3 introduces more complex law of motion with additional lags on top of Model 2. It is only necessary to specify Model 3 because Model 1 and 2 are special cases of it. We denote Δu_t as $\Delta UNEMP_t$ and Δr_t as $\Delta tb3m_t$. The matrix form of the actual law of motion satisfies

$$\begin{bmatrix} \Delta r_t \\ \Delta u_t \end{bmatrix} = \begin{bmatrix} 0 & \phi_{ru0} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta r_t \\ \Delta u_t \end{bmatrix} + \begin{bmatrix} \phi_{rr1} & 0 \\ \phi_{ur1} & \phi_{uu1} \end{bmatrix} \begin{bmatrix} \Delta r_{t-1} \\ \Delta u_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{rr2} & 0 \\ 0 & \phi_{uu2} \end{bmatrix} \begin{bmatrix} \Delta r_{t-2} \\ \Delta u_{t-2} \end{bmatrix} + \begin{bmatrix} \sigma_r & 0 \\ 0 & \sigma_u \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \tag{40}$$

Note that a short-run restriction that assumes interest rate has no contemporaneous effects on unemployment rate has been imposed here. Some co-movements of these two variables are also shut down to avoid too many parameters in estimation. The stacked reduced form of the structural VAR model is

$$\underbrace{\begin{bmatrix} \Delta r_t \\ \Delta u_t \\ \Delta r_{t-1} \\ \Delta u_{t-1} \end{bmatrix}}_{\mathbf{x}_t} = \underbrace{\begin{bmatrix} \phi_{rr1} + \phi_{ru0}\phi_{ur1} & \phi_{ru0}\phi_{uu1} & \phi_{rr2} & \phi_{ru0}\phi_{uu2} \\ \phi_{ur1} & \phi_{uu1} & 0 & \phi_{uu2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{\Phi}} \underbrace{\begin{bmatrix} \Delta r_{t-1} \\ \Delta u_{t-1} \\ \Delta r_{t-2} \\ \Delta u_{t-2} \end{bmatrix}}_{\mathbf{x}_{t-1}} + \underbrace{\begin{bmatrix} \sigma_r & \phi_{ru0}\sigma_u & 0 & 0 \\ 0 & \sigma_u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{\epsilon}_t}. \tag{41}$$

We also have $\mathbf{y}_t = \mathbf{\Psi}\mathbf{x}_t$, in which

$$\mathbf{y}_t = \begin{bmatrix} \Delta r_t \\ \Delta u_t \end{bmatrix}, \qquad \mathbf{\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{42}$$

Forecasters receive Gaussian noisy signals about period t macroeconomic indicators at the same period but perfectly observe all public releases of data in the past.[9] All forecasters share the same parameters but differ in the ex-post private signal noise they receive.

Proposition 4 (Kalman gain)**.** *In the model of Section 4.3,*

$$\widetilde{\mathbf{y}}_{t+h|t} = \mathbf{\Psi}\widetilde{\mathbf{\Phi}}^h[\widetilde{\mathcal{K}}\mathbf{x}_t + (\mathbf{1} - \widetilde{\mathcal{K}})\widetilde{\mathbf{\Phi}}\mathbf{x}_{t-1}], \tag{43}$$

in which $\widetilde{\mathcal{K}} \equiv \widetilde{\mathbf{\Sigma}}(\widetilde{\mathbf{\Sigma}} + \widetilde{\mathbf{\Omega}})^{-1}$.

Proof. See Section A.9. □

[9]This assumption is consistent with the nearly perfect 1-quarter backcast of professional forecasters. This assumption also greatly simplify the information friction by eliminating the persistent information.

As a result, this model has 18 parameters in total. The actual law of motion is captured by $\{\phi_{ru0}, \phi_{rr1}, \phi_{ur1}, \phi_{uu1}, \phi_{rr2}, \phi_{uu2}, \sigma_r, \sigma_u\}$. The perceived model is characterized by the perceived transition matrix elements $\{\widetilde{\phi}_{ru0}, \widetilde{\phi}_{rr1}, \widetilde{\phi}_{ur1}, \widetilde{\phi}_{uu1}, \widetilde{\phi}_{rr2}, \widetilde{\phi}_{uu2}\}$, and Kalman gain matrix elements $\widetilde{\mathcal{K}}(j,k)$ for $\forall j,k \in \{1,2\}$. Model 3 imposes no restriction on parameters. Model 2 imposes only $(\widetilde{\phi}_{rr2}, \widetilde{\phi}_{uu2}) = (\phi_{rr2}, \phi_{uu2}) = (0,0)$. Model 1 further imposes $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}$.

Estimation. The parameters are estimated to match $60 = 3 \times 5 \times 4$ covariance statistics, which include the following 3 types of covariance statistics

$$\{cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k)), \quad cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k)) - cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k)),$$
$$cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k)) - cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_t(k))\}, \quad (44)$$

for 5 different horizons $h \in \{0, 1, \cdots, 4\}$ and the 4 types of variable combinations below

$$(\mathbf{y}(j), \mathbf{y}(k)) \in \{(-\Delta UNEMP, -\Delta UNEMP), \quad (-\Delta UNEMP, \Delta tb3m),$$
$$(\Delta tb3m, -\Delta UNEMP), \quad (\Delta tb3m, \Delta tb3m)\}. \quad (45)$$

$cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k))$ is the autocovariance of outcomes, $cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k)) - cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k))$ captures the perceived information friction, and $cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k)) - cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_t(k))$ describes the misperceived law of motion. Proposition 5 below ensures that these moments are informative for the the empirical performance of the VAR model.

Proposition 5 (Informative statistics)**.** *In the model of Section 4.3,*

1. *The collection of* $\{cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k))\}$ *for* $\forall j,k \in \mathbb{N}_{>0}$ *and* $\forall h \in \mathbb{N}$ *fully characterizes the process of* $\{\mathbf{y}_t\}$*.*
2. *When* $\widetilde{\mathbf{\Omega}}_{i,\ell} = \mathbf{0}$, $cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k)) = cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k))$ *for* $\forall j,k \in \mathbb{N}_{>0}$ *and* $\forall h \in \mathbb{N}$*.*
3. *When* $\widetilde{\Xi}_i = \Xi$, $cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k)) = cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_t(k))$ *for* $\forall j,k \in \mathbb{N}_{>0}$ *and* $\forall h \in \mathbb{N}$*.*

Proof. See Section A.10. □

The parameters are estimated using simulated method of moments. Let $\mathbf{m}_{model}(\boldsymbol{\beta})$ and $\mathbf{m}_{data}$ denote the target moments in both model and data, in which $\boldsymbol{\beta}$ is the set of parameters to be estimated. The estimation procedures searches $\boldsymbol{\beta}$ to minimize the following function.

$$f(\boldsymbol{\beta}) \equiv (\mathbf{m}_{data} - \mathbf{m}_{model}(\boldsymbol{\beta}))^\intercal \mathbf{W} (\mathbf{m}_{data} - \mathbf{m}_{model}(\boldsymbol{\beta})),$$

in which $\mathbf{W}$ is the optimal weight that captures the standard errors of $\mathbf{m}_{data}$. In theory, $\mathbf{W}$ is the inverse covariance matrix of $\mathbf{m}_{data}$. In practice, we treat $\mathbf{m}_{data}$ as i.i.d. random variables. The standard errors of $\boldsymbol{\beta}$ is computed via a Delta method.

The estimation is conducted in two steps. First, I estimate the parameters on the actual law of motion to target $cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k))$. Second, I estimate the other parameters to target the remaining moments, treating the point estimate of the first step as the true values. This procedure makes sure that the actual law of motion is identified only by the data variations of the actual outcomes $\{\mathbf{y}_t\}$. The estimated parameter values are summarized in Table 4.

[Table 4 about here.]

Performance. The performance of the following three types of statistics are reported in Table 5.

$$\left\{ cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k)),\ \frac{cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k))}{cov(\mathbf{y}_{t+h}(j), \mathbf{y}_t(k))},\ \frac{cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_t(k))}{cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k))} \right\}$$
$$\equiv \{z_h^{auto}(j,k),\ z_h^{state}(j,k),\ z_h^{term}(j,k)\}, \quad (46)$$

which measures the autocovariance of the actual outcomes, the perceived information friction, and the term structure of misperceived law of motion, according to Proposition 5.

The moment $z_h^{auto}(j,k)$ in Model 1 and Model 2 matches data quite well. The extra lags in Model 3 improve the model performance very little. These results indicate that a simple VAR(1) model is sufficiently rich to capture the main properties of the actual law of motion, which should be easy to understand by professional forecasters.

The moments of $z_h^{state}(j,k)$ in all models also match data closely. The results imply that a simple information friction characterized by Proposition 4 is sufficiently rich to capture the patterns of the misperceived nowcasts.

[Table 5 about here.]

The main takeaways are in $z_h^{term}(j,k)$. Model 2 improves the performance of $z_h^{term}(j,k)$ to a large extent by relaxing the restriction $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}$. $z_h^{term}(j,k)$ in the model becomes strictly decreasing in h like the data but never goes negative when $h = 4$. Model 3 allows $z_h^{term}(j,k)$ with $tb3m$ to go negative when $h = 4$, but the absolute values of $z_h^{term}(j,k)$ are still smaller than what is required to match data. This puzzling pattern is highlighted in Figure 5.

[Figure 5 about here.]

The difficulty of rationalizing $z_h^{term}(j,k)$ for $tb3m$ despite the ease of explaining $z_h^{auto}(j,k)$ suggests that the professional forecasters likely overcomplicate the law of motion of interest rate dynamics.[10]

5 Conclusion

This paper proposes a new empirical method to detect misperceived law of motion using survey expectations within a wide range of linear Gaussian expectation formation models. A key feature of the method is that it does not rely on any prior knowledge of the specific model structure. This feature allows the method to provide decisive evidence for misperceived laws of motion, ruling out alternative explanations for the same evidence. The only assumption required is a signal extraction problem that generates subjective expectations with potentially incorrect parameter values.

The minimal data requirement of this new method is the time series of two consensus expectations along with their actual outcomes. This feature makes the method applicable

[10]The model of Farmer et al. (2023) does a better job in replicating $z_4^{term}(j,k) < 0$ than the VAR model in this section, but has problems in replicating $z_h^{term}(j,k) > 0$ for $h \leq 2$.

to data from the Michigan Survey of Consumers, facilitating the incorporation of consumer expectations into business cycle analysis.

This paper presents three applications to demonstrate the use of the proposed method to understand anomalies in the expectations among consumers and professional forecasters. Application 1 reveals that although consumers observe the autocorrelation of output growth in the past, they fail to incorporate this knowledge when forecasting future growth. Application 2 suggests that while the lack of attention on supply shocks can generate the negative correlation between inflation and growth forecasts among consumers, the lack of knowledge on how supply shocks affect inflation is needed to further explain why inflation forecasts and growth outcomes are also negative correlated. Applications 3 demonstrates that while the actual law of motion for the 3-month treasury rate can be well captured by a VAR model, the evidence for misperceived law of motion is difficult to explain within a misperceived VAR. As a summary, consumers tend to oversimplify the dynamics of growth and inflation, while professional forecasters tend to overcomplicate the dynamics of short-term interest rates.

Furthermore, a promising application beyond the scope of this paper involves using this method with consumer expectations data to discipline a behavioral DSGE model. This line of work has been challenging due to the lack of decisive evidence in previous studies.[11] The new method in this paper has the potential to advance the frontier in this direction.

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[11] Qiu (2019); Bhandari et al. (2019); Pei (2023) have made some contribution in this line.

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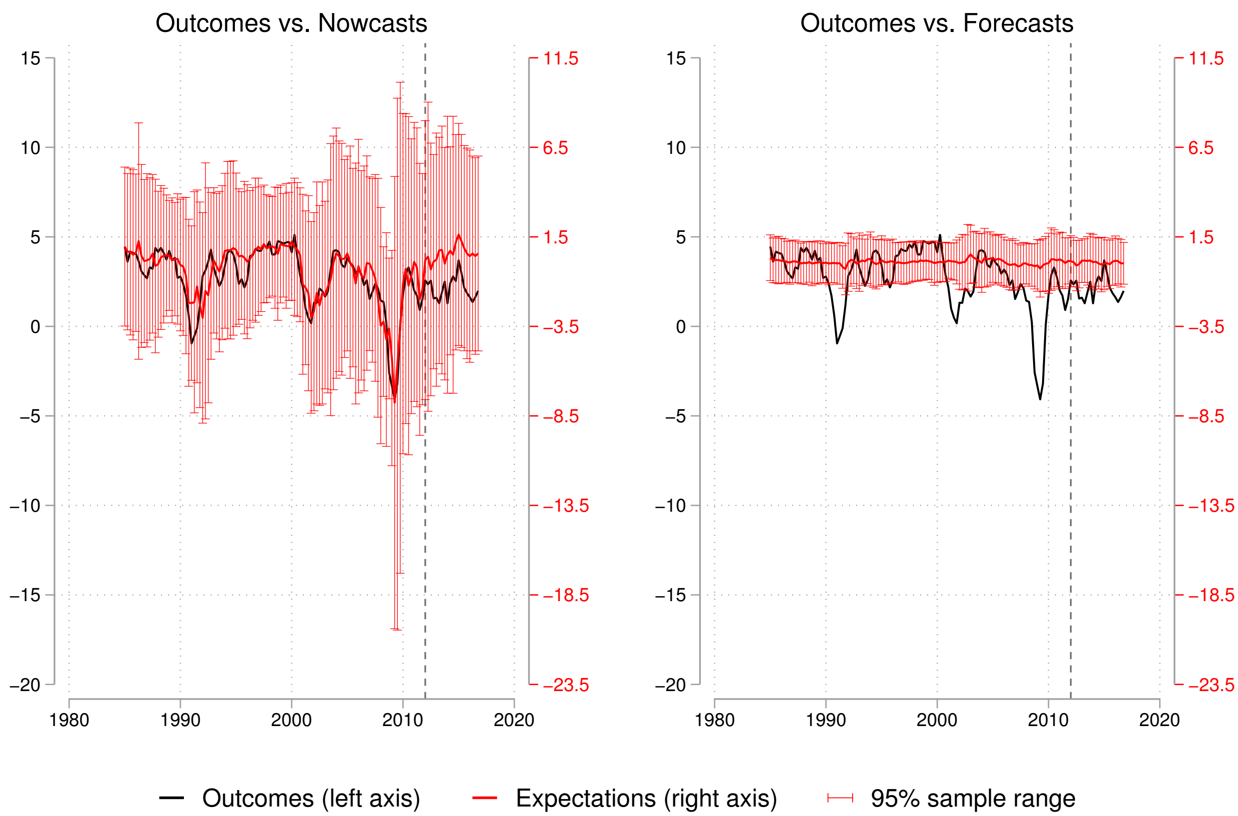

Figure 1: Outcomes vs Transformed Expectations of the Real GDP Growth

Notes: The nowcasts and forecasts are converted from the categorical nowcast and forecast data on the business cycle changes in a year from the Michigan Survey of Consumers, following the method of Mankiw et al. (2003). Both consensus levels and the dispersion are depicted in Figure 1. The outcomes are the corresponding annual real GDP growth from FRED. This figure shows that the consensus nowcast is nearly perfectly correlated with the outcomes, while the consensus forecast has little variation over time.

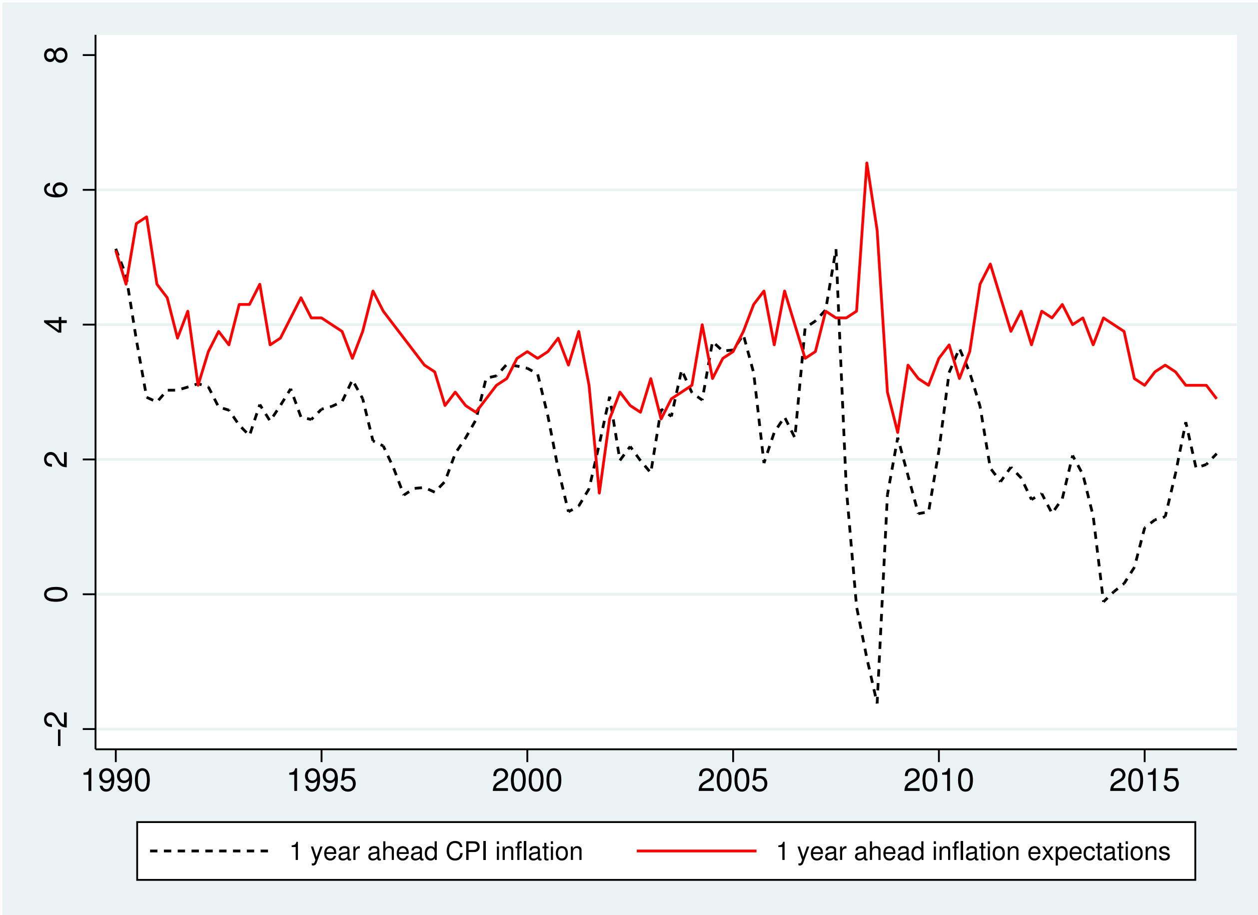

Figure 2: Disconnection of Inflation Expectations to the Actual CPI Inflation

Notes: In Figure 2, the 1-year ahead inflation expectations is the sample mean the counterpart in the Michigan Survey of Consumers, while the 1-year ahead CPI inflation is from FRED.

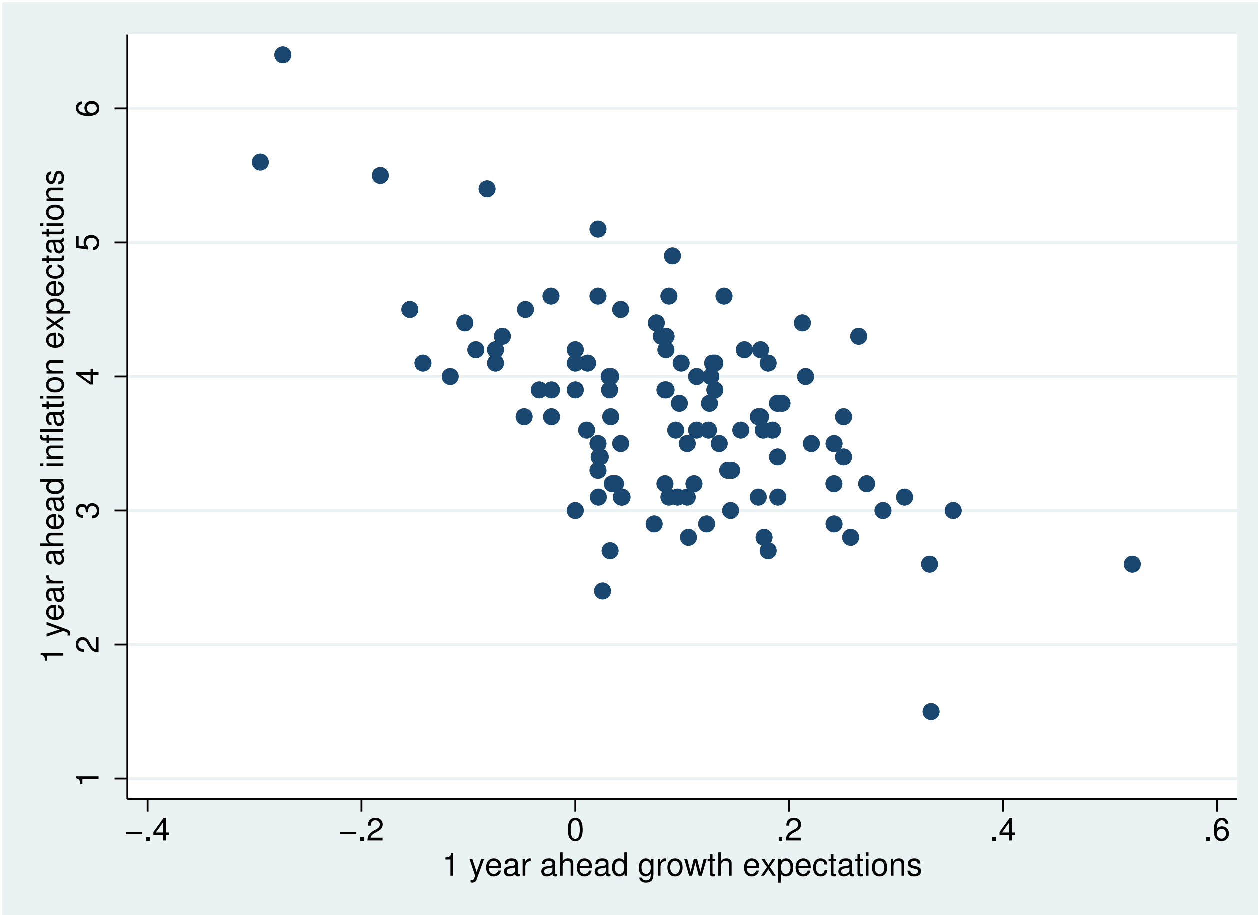

Figure 3: Misaligned Inflation Expectations

Notes: Figure 3 shows the negative correlation between inflation forecasts and growth forecasts for Section 4.2. The 1-year ahead inflation expectations is the sample mean of the counterpart in the Michigan Survey of Consumers, while the 1-year ahead growth expectations is the same as the growth forecasts in Figure 1. The sample range is 1990-2016.

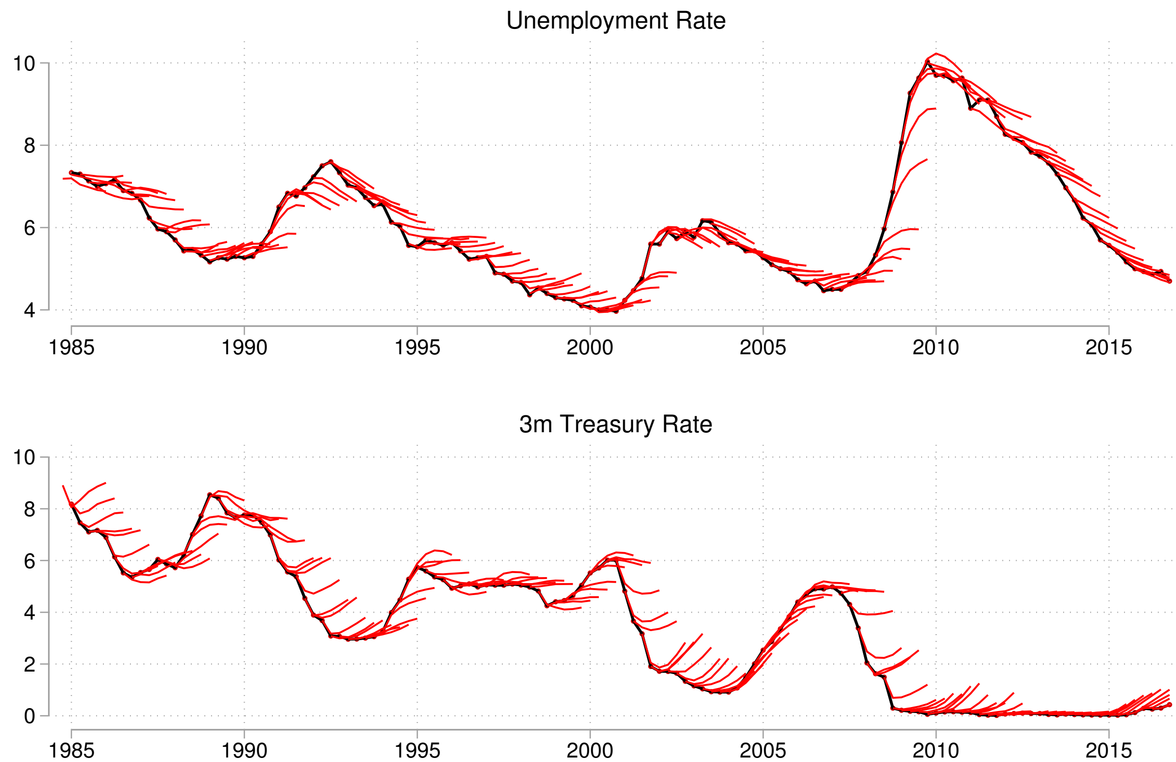

Figure 4: Expectations and Outcomes Across Forecast Horizons

Notes: In Figure 4, the long solid lines represent the actual outcomes of the chosen indicators in Section 4.3, while the short solid lines connected with the long lines represent the corresponding consensus expectations in the Survey of Professional Forecasters at horizon $h \in \{-1, 0, \cdots, 4\}$.

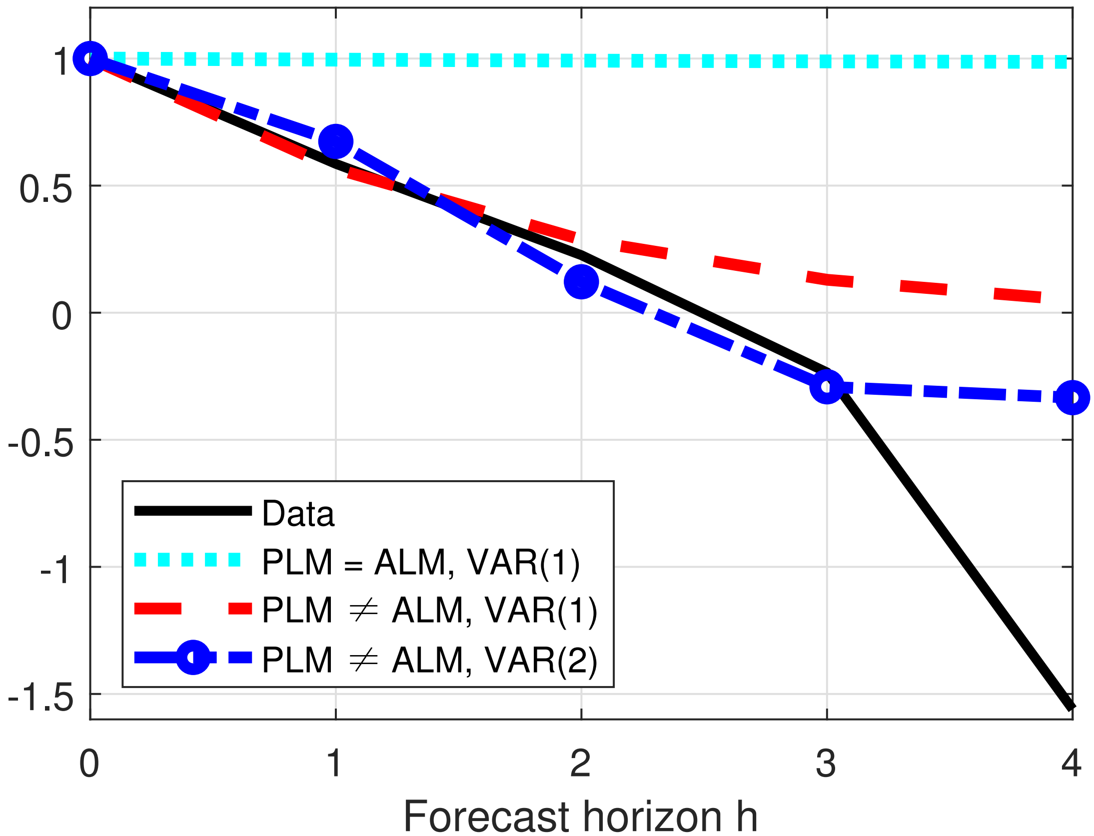

Figure 5: Term Structure of $z_h^{term}(j,k)$ for $(\mathbf{y}(j), \mathbf{y}(k)) = (\Delta tb3m, \Delta tb3m)$

Notes: Figure 5 compares the statistics $z_h^{term}(j,k)$ that capture the misperceived law of motion across forecast horizons in the data with their counterparts in three models, which correspond to Model 1-3 in Section 4.3. The results indicate that the statistic $z_h^{term}(j,k)$ at horizon $h = 4$ is difficult to rationalize using misperceived VAR models.

Table 1: Naive Growth Forecasts in the Michigan Survey of Consumers

Subsample	z^{auto}	z^{now}	z^{fore}	$z^{auto} - z^{now}$	$z^{now} - z^{fore}$	z^{now}/z^{auto}	z^{fore}/z^{now}
All	0.981	1.089	0.012	-0.108	1.078	1.110	0.011
	(0.400)	(0.571)	(0.020)	(0.234)	(0.566)		
				By Age			
18-34		1.377	0.007	-0.396	1.369	1.404	0.005
		(0.732)	(0.020)	(0.390)	(0.729)		
35-44		1.490	0.023	-0.550	1.474	1.519	0.015
		(0.857)	(0.022)	(0.547)	(0.853)		
45-54		1.633	0.023	-0.652	1.610	1.665	0.014
		(0.971)	(0.028)	(0.626)	(0.953)		
55-64		1.588	-0.008	-0.607	1.595	1.619	-0.005
		(0.968)	(0.022)	(0.624)	(0.955)		
65-97		0.807	-0.006	0.174	0.812	0.823	-0.007
		(0.453)	(0.031)	(0.165)	(0.465)		
				By Region			
West		1.096	0.035	-0.115	1.060	1.117	0.032
		(0.646)	(0.025)	(0.292)	(0.627)		
N center		1.383	-0.003	-0.403	1.386	1.410	-0.002
		(0.784)	(0.024)	(0.444)	(0.782)		
N east		1.405	0.001	-0.424	1.404	1.432	0.001
		(0.710)	(0.033)	(0.363)	(0.714)		
South		1.241	0.017	-0.260	1.224	1.265	0.014
		(0.703)	(0.026)	(0.329)	(0.690)		
				By Gender			
Male		1.157	-0.003	-0.176	1.159	1.179	-0.003
		(0.601)	(0.031)	(0.269)	(0.582)		
Female		1.417	0.022	-0.436	1.396	1.444	0.016
		(0.869)	(0.020)	(0.507)	(0.873)		
				By Income			
1st 25%		1.484	0.022	-0.503	1.462	1.513	0.015
		(1.011)	(0.026)	(0.650)	(1.023)		
2nd 25%		1.094	0.029	-0.113	1.065	1.115	0.027
		(0.614)	(0.019)	(0.272)	(0.608)		
3rd 25%		1.198	0.018	-0.259	1.192	1.221	0.015
		(0.613)	(0.030)	(0.290)	(0.604)		
4th 25%		1.269	-0.031	-0.406	1.307	1.294	-0.024
		(0.632)	(0.039)	(0.378)	(0.624)		
				By Education			
High		1.224	0.029	-0.243	1.196	1.248	0.024
		(0.770)	(0.014)	(0.408)	(0.771)		
Some Col		1.128	0.041	-0.147	1.086	1.150	0.036
		(0.558)	(0.028)	(0.218)	(0.536)		
Col degree		1.416	-0.018	-0.476	1.441	1.443	-0.013
		(0.740)	(0.027)	(0.439)	(0.740)		
Graduate		1.380	-0.050	-0.399	1.430	1.407	-0.036
		(0.693)	(0.041)	(0.351)	(0.696)		

Notes: Table 2 provides the informative statistics for Section 4.1. In particular, the ratio z^{fore}/z^{now}, which is supposed to be one when perceived and actual laws of motion align, is zero across all subsamples. The results indicate that all type of consumers share similar naive understanding on the autocorrelation of output growth. The standard errors in parentheses are Newey-West with the automatic bandwidth selection approach in Newey and West (1994).

Table 2: Misaligned Inflation Expectations in the Michigan Survey of Consumers

	$\frac{cov(\pi_{t+4}, g_{t+4})}{std(\pi_{t+4})std(g_{t+4})}$	$\frac{cov(\widetilde{\pi}_{t+4\|t}, \widetilde{g}_{t+4\|t})}{std(\pi_{t+4})std(g_{t+4})}$	$\frac{cov(\widetilde{\pi}_{t+4\|t}, g_{t+4}) - cov(\pi_{t+4}, \widetilde{g}_{t+4\|t})}{std(\pi_{t+4})std(g_{t+4})}$
	0.215 (0.265)	-0.028 (0.010)	-0.267 (0.139)
obs	108	108	108

Notes: Table 1 provides the informative statistics for the misaligned inflation expectations in Section 4.2. In particular, the normalized comovement between inflation and growth is weakly positive, while the comovement between inflation forecasts and growth forecasts and that between inflation forecasts and growth outcomes tend to be negative. The standard errors in the parentheses are Newey-West with the automatic bandwidth selection approach in Newey and West (1994).

Table 3: Testing $cov(y_{t+h}, \widetilde{y}_{t|t}) - cov(\widetilde{y}_{t+h|t}, y_t) = 0$ in Section 4.3

h	$y = \Delta UNEMP$	$y = \Delta tb3m$
1	0.005	0.032
	(0.005)	(0.011)
2	0.016	0.038
	(0.005)	(0.011)
3	0.014	0.041
	(0.005)	(0.015)
4	0.019	0.026
	(0.004)	(0.013)
obs	128	128

Notes: Table 3 provides a list of statistics in Section 4.3 for the unemployment rate changes and the 3-moth treasury rate changes. Most of the statistics can detect misperceived law of motion across multiple forecast horizons of $h \in \{1, 2, 3, 4\}$ following the logic of Theorem 1. The standard errors in parentheses are Newey-West with the automatic bandwidth selection approach in (Newey and West, 1994).

Table 4: Estimated Parameters for Section 4.3

	Model 1		Model 2		Model 3	
Parameter	Actual	Perceived	Actual	Perceived	Actual	Perceived
ϕ_{ru0} or $\widetilde{\phi}_{ru0}$	0.290	0.290	0.290	-0.075	0.281	0.073
	(0.142)		(0.142)	(0.203)	(0.145)	(0.258)
ϕ_{rr1} or $\widetilde{\phi}_{rr1}$	0.503	0.503	0.503	0.349	0.648	0.626
	(0.116)		(0.116)	(0.100)	(0.342)	(0.130)
ϕ_{ur1} or $\widetilde{\phi}_{ur1}$	0.112	0.112	0.112	0.123	0.141	0.134
	(0.065)		(0.065)	(0.051)	(0.081)	(0.056)
ϕ_{uu1} or $\widetilde{\phi}_{uu1}$	0.640	0.640	0.640	0.432	0.265	0.364
	(0.098)		(0.098)	(0.073)	(0.301)	(0.219)
ϕ_{rr2} or $\widetilde{\phi}_{rr2}$					-0.110	-0.237
					(0.244)	(0.114)
ϕ_{uu2} or $\widetilde{\phi}_{uu2}$					0.300	0.007
					(0.253)	(0.147)
σ_r	0.327		0.327		0.295	
	(0.051)		(0.051)		(0.084)	
σ_u	0.199		0.199		0.239	
	(0.050)		(0.050)		(0.058)	
$\widetilde{\mathcal{K}}(1,1)$		0.666		0.721		0.640
		(0.066)		(0.056)		(0.070)
$\widetilde{\mathcal{K}}(1,2)$		-0.277		-0.061		-0.029
		(0.078)		(0.079)		(0.062)
$\widetilde{\mathcal{K}}(2,1)$		-0.081		-0.089		-0.104
		(0.038)		(0.038)		(0.047)
$\widetilde{\mathcal{K}}(2,2)$		0.326		0.580		0.611
		(0.151)		(0.120)		(0.126)

Notes: Table 4 provides the estimation results of the 18 parameters in the models of Section 4.3. The standard errors in parentheses are obtained from simulated method of moments with the optimal weighting matrix. The results of Model 1 indicate that the estimated Kalman gain $\widetilde{\mathcal{K}}(2,2)$ for the 3-month treasury rate is smaller if the perceived law of motion is restricted to be identical to the actual one. The results of Model 2 indicate that the estimated perceived connections of the 3-month treasury rate changes with the contemporary unemployment rate changes $\widetilde{\phi}_{ru0}$ is underestimated by the professional forecasters. Professional forecasters appear to misunderstand the Taylor Rule to some extent.

Table 5: The Match between Model and Data for Section 4.3

		$z_h^{auto}(j,k)$			$z_h^{state}(j,k)$				$z_h^{term}(j,k)$		
h	Data	M1&2	M3	Data	M1	M2	M3	Data	M1	M2	M3
					$(\mathbf{y}(j),\mathbf{y}(k)) = (-\Delta UNEMP, -\Delta UNEMP)$						
0	0.084	0.083	0.089	0.645	0.667	0.703	0.677	1.000	1.000	1.000	1.000
1	0.047	0.059	0.044	0.682	0.667	0.693	0.739	0.843	1.005	0.732	0.843
2	0.044	0.042	0.044	0.711	0.666	0.686	0.685	0.495	1.009	0.488	0.413
3	0.030	0.031	0.030	0.696	0.666	0.682	0.703	0.322	1.011	0.307	0.235
4	0.026	0.023	0.025	0.702	0.666	0.679	0.688	-0.038	1.012	0.185	0.090
					$(\mathbf{y}(j),\mathbf{y}(k)) = (-\Delta UNEMP, \Delta tb3m)$						
0	0.055	0.050	0.052	0.792	0.702	0.745	0.756	0.840	0.948	0.792	0.811
1	0.042	0.051	0.049	0.689	0.727	0.753	0.779	0.716	0.969	0.750	0.746
2	0.046	0.044	0.044	0.698	0.736	0.757	0.758	0.273	0.976	0.536	0.538
3	0.035	0.035	0.036	0.602	0.740	0.758	0.762	0.028	0.979	0.352	0.278
4	0.029	0.027	0.028	0.582	0.742	0.759	0.757	-0.335	0.981	0.221	0.080
					$(\mathbf{y}(j),\mathbf{y}(k)) = (\Delta tb3m, -\Delta UNEMP)$						
0	0.055	0.050	0.052	0.665	0.665	0.589	0.613	1.191	1.055	1.263	1.233
1	0.043	0.042	0.041	0.542	0.666	0.631	0.616	0.537	1.035	0.403	0.597
2	0.030	0.033	0.033	0.638	0.666	0.651	0.643	0.121	1.025	0.123	0.046
3	0.025	0.026	0.025	0.710	0.666	0.662	0.667	-0.253	1.020	0.026	-0.154
4	0.020	0.019	0.020	0.643	0.666	0.668	0.679	-0.656	1.017	-0.004	-0.130
					$(\mathbf{y}(j),\mathbf{y}(k)) = (\Delta tb3m, \Delta tb3m)$						
0	0.166	0.172	0.165	0.811	0.768	0.768	0.756	1.000	1.000	1.000	1.000
1	0.105	0.102	0.109	0.723	0.762	0.766	0.743	0.586	0.996	0.566	0.674
2	0.065	0.064	0.065	0.759	0.757	0.764	0.742	0.226	0.992	0.287	0.122
3	0.050	0.042	0.040	0.660	0.753	0.763	0.747	-0.235	0.989	0.130	-0.291
4	0.021	0.029	0.027	0.472	0.750	0.762	0.751	-1.560	0.987	0.050	-0.334

Notes: Table 5 provides the comparison between the three models and the data in terms of the informative statistics defined in equation (46) for the application in Section 4.3. In particular, M1, M2, and M3 are short for Model 1, Model 2, and Model 3, respectively. We have a special focus on $z_h^{term}(j,k)$, which are supposed to be one if perceived and actual laws of motion align. The results indicate that the negative value of $z_h^{term}(j,k)$ at the horizons of $h = 4$ is difficult to rationalize in a misperceived VAR model, especial when the pair of variables (j,k) are both the 3-month treasury rate.

Appendix for Online Publication

A Omitted Proofs

A.1 Proof of Lemma 1

Imposing the subjective expectation operator $\widetilde{\mathbb{E}}_{i,t}$ on the actual law of motion $\mathbf{x}_{t+h'} = \mathbf{\Phi}\mathbf{x}_{t-1+h'} + \boldsymbol{\epsilon}_{t+h'}$ for $h' \in \{1, 2, \cdots, h\}$ in Example 1 yields

$$\widetilde{\mathbf{x}}_{i,t+h'|t} = \widetilde{\mathbf{\Phi}} \cdot \widetilde{\mathbf{x}}_{i,t-1+h'|t} \implies \widetilde{\mathbf{x}}_{i,t+h|t} = \widetilde{\mathbf{\Phi}}^h \cdot \widetilde{\mathbf{x}}_{i,t|t} \implies \widetilde{\mathbf{y}}_{t+h|t} = \widetilde{\mathbf{\Phi}}^h \cdot \widetilde{\mathbf{y}}_{t|t}. \tag{47}$$

A.2 Proof of Proposition 1

$$\frac{\widetilde{\mathbf{\Phi}}^h \cdot cov(\widetilde{\mathbf{y}}_{t|t}, \mathbf{y}_t)}{\mathbf{\Phi}^h \cdot cov(\mathbf{y}_t, \widetilde{\mathbf{y}}_{t|t})} = \frac{cov(\widetilde{\mathbf{\Phi}}^h \widetilde{\mathbf{y}}_{t|t}, \mathbf{y}_t)}{cov(\mathbf{\Phi}^h \mathbf{y}_t + \boldsymbol{\epsilon}_t, \widetilde{\mathbf{y}}_{t|t})} = \frac{cov(\widetilde{\mathbf{y}}_{t+h|t}, \mathbf{y}_t)}{cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{t|t})}. \tag{48}$$

A.3 Proof of Theorem 1

Due to the covariance stationarity, for $\forall t \in \mathbb{Z}$, we can denote the power spectral density as

$$\widetilde{\mathbf{R}}_{i,\mathbf{y}(j)\mathbf{w}}(\tau) \equiv cov(\mathbf{y}_{t+h}(j), \mathbf{s}_{i,\tau,t} | \widetilde{\mathbf{\Theta}}_i), \tag{49}$$

$$\widetilde{\mathbf{R}}_{i,\mathbf{w}\mathbf{y}(k)}(\tau) \equiv cov(\mathbf{s}_{i,\tau,t}, \mathbf{y}_{t+h}(k) | \widetilde{\mathbf{\Theta}}_i), \tag{50}$$

$$\widetilde{\mathbf{R}}_{i,\mathbf{ww}}(\tau) \equiv cov(\mathbf{s}_{i,0,t}, \mathbf{s}_{i,\tau,t} | \widetilde{\mathbf{\Theta}}_i). \tag{51}$$

The corresponding power spectral density functions are

$$\widetilde{\mathbf{S}}_{i,\mathbf{y}(j)\mathbf{w}}(z) \equiv \sum_{\tau=-\infty}^{+\infty} \widetilde{\mathbf{R}}_{i,\mathbf{y}(j)\mathbf{w}}(\tau) z^{-\tau}, \tag{52}$$

$$\widetilde{\mathbf{S}}_{i,\mathbf{w}\mathbf{y}(k)}(z) \equiv \sum_{\tau=-\infty}^{+\infty} \widetilde{\mathbf{R}}_{i,\mathbf{w}\mathbf{y}(k)}(\tau) z^{-\tau}, \tag{53}$$

$$\widetilde{\mathbf{S}}_{i,\mathbf{ww}}(z) \equiv \sum_{\tau=-\infty}^{+\infty} \widetilde{\mathbf{R}}_{i,\mathbf{ww}}(\tau) z^{-\tau}. \tag{54}$$

Only consider the generic case when $\widetilde{\mathbf{S}}_{i,\mathbf{ww}}(z)^{-1}$ always exists. It can be decomposed as

$$\widetilde{S}_{i,\mathbf{ww}}(z)^{-1} = \widetilde{\mathbf{B}}_i(z^{-1})^T\widetilde{\mathbf{B}}_i(z), \quad \text{in which} \quad \widetilde{\mathbf{B}}_i(z) = \sum_{\tau=0}^{+\infty} \widetilde{B}_{i,\tau} z^{-\tau}. \tag{55}$$

Let $[\cdot]_+$ denotes the causal part of a spectral density, we have

$$\left[\widetilde{\mathbf{S}}_{i,\mathbf{y}(j)\mathbf{w}}(z)\widetilde{\mathbf{B}}_i^T(z^{-1})\right]_+ = \sum_{\tau=0}^{+\infty}\sum_{\nu=0}^{+\infty} \widetilde{\mathbf{R}}_{i,\mathbf{y}(j)\mathbf{w}}(\tau+\nu)\widetilde{\mathbf{B}}_{i,\nu}^T z^{-\tau}, \tag{56}$$

$$\widetilde{\mathbf{B}}_i(z)\mathbf{S}_{i,\mathbf{wy}(k)}(z^{-1}) = \sum_{\tau=-\infty}^{+\infty}\sum_{\nu=0}^{+\infty} \widetilde{\mathbf{B}}_{i,\nu}\mathbf{R}_{i,\mathbf{wy}(k)}(-\tau-\nu)z^{\tau}. \tag{57}$$

Using **Wiener-Hopf formular**,

$$\mathbf{S}_{i,\widetilde{\mathbf{y}}(j)\mathbf{y}(k)}(z) = \left[\widetilde{\mathbf{S}}_{i,\mathbf{y}(j)\mathbf{w}}(z)\widetilde{\mathbf{B}}_i^T(z^{-1})\right]_+ \widetilde{\mathbf{B}}_i(z)\mathbf{S}_{i,\mathbf{wy}(k)}(z^{-1}). \tag{58}$$

The inverse z-transform yields

$$\begin{aligned} &cov(\widetilde{\mathbf{y}}_{i,t+h|t}(j), \mathbf{y}_{t+h}(k)) = \int_{-\pi}^{+\pi} \mathbf{S}_{i,\widetilde{\mathbf{y}}(j)\mathbf{y}(k)}(e^{j\omega})\frac{d\omega}{2\pi} = \mathbf{S}_{i,\widetilde{\mathbf{y}}(j)\mathbf{y}(k)}(0) \\ &\quad = \sum_{\tau=0}^{+\infty}\left[\sum_{\nu=0}^{+\infty}\widetilde{\mathbf{R}}_{i,\mathbf{y}(j)\mathbf{w}}(\tau+\nu)\widetilde{\mathbf{B}}_{i,\nu}^T\right]\left[\sum_{\nu=0}^{+\infty}\widetilde{\mathbf{B}}_{i,\nu}\mathbf{R}_{i,\mathbf{wy}(k)}(-\tau-\nu)\right], \\ &\quad = \sum_{\tau=0}^{+\infty}\left(\sum_{\nu=0}^{+\infty} cov(\mathbf{y}_{t+h}(j), \mathbf{s}_{i,\tau+\nu,t}|\widetilde{\Xi}_i)\widetilde{\mathbf{B}}_{i,\nu}^T\right)\left(\sum_{\nu=0}^{+\infty}\widetilde{\mathbf{B}}_{i,\nu} cov(\mathbf{s}_{i,\tau+\nu,t}, \mathbf{y}_{t+h}(k)|\Xi_i)\right). \end{aligned} \tag{59}$$

Similarly, the other side is

$$\begin{aligned} &cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{i,t+h|t}(k)) \\ &\quad = \sum_{\tau=0}^{+\infty}\left(\sum_{\nu=0}^{+\infty} cov(\mathbf{y}_{t+h}(j), \mathbf{s}_{i,\tau+\nu,t}|\Xi_i)\widetilde{\mathbf{B}}_{i,\nu}^T\right)\left(\sum_{\nu=0}^{+\infty}\widetilde{\mathbf{B}}_{i,\nu} cov(\mathbf{s}_{i,\tau+\nu,t}, \mathbf{y}_{t+h}(k)|\widetilde{\Xi}_i)\right). \end{aligned} \tag{60}$$

Now, it is obvious that $\widetilde{\Xi}_i = \Xi$ implies

$$cov(\widetilde{\mathbf{y}}_{i,t+h|t}(j), \mathbf{y}_{t+h}(k)) = cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{i,t+h|t}(k)). \tag{61}$$

Aggregation over individual expectations yields

$$
\begin{aligned}
cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_{t+h}(k)) &= \frac{1}{M}\sum_{i=1}^{M} cov(\widetilde{\mathbf{y}}_{i,t+h|t}(j), \mathbf{y}_{t+h}(k)) \\
&= \frac{1}{M}\sum_{i=1}^{M} cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{i,t+h|t}(k)) = cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t+h|t}(k)). \quad (62)
\end{aligned}
$$

Note that the impacts of perceived information frictions are all in $\{\widetilde{\mathbf{B}}_{i,v}\}$, which is controlled in the comparison between $cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_t(k))$ and $cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t|t}(k))$.

A.4 Proof of Theorem 2

Under rational expectations, individual expectation errors are unpredictable by individual information set, so that

$$
cov(\widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h} - \widetilde{\mathbf{y}}_{i,t+h|t}) = cov(\mathbf{y}_{t+h} - \widetilde{\mathbf{y}}_{i,t+h|t}, \widetilde{\mathbf{y}}_{i,t+h|t}) = 0, \quad (63)
$$

As a result, we can derive

$$
\begin{aligned}
& M \cdot cov(\widetilde{\mathbf{y}}_{t+h|t}, \mathbf{y}_{t+h}) = cov(\textstyle\sum_{i=1}^{M} \widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h}) \\
&= \textstyle\sum_{i=1}^{M} cov(\widetilde{\mathbf{y}}_{i,t+h|t}, \mathbf{y}_{t+h}) = \sum_{i=1}^{M} cov(\widetilde{\mathbf{y}}_{i,t+h|t}, \widetilde{\mathbf{y}}_{i,t+h|t}) = \sum_{i=1}^{M} cov(\mathbf{y}_{t+h}, \widetilde{\mathbf{y}}_{i,t+h|t}) \\
&\qquad = \mathbf{y}_{t+h}, cov(\textstyle\sum_{i=1}^{M} \widetilde{\mathbf{y}}_{i,t+h|t}) = M \cdot cov(\mathbf{y}_{t+h|t}, \widetilde{\mathbf{y}}_{t+h|t}). \quad (64)
\end{aligned}
$$

A.5 Proof of Corollary 1

$\frac{1}{T}\sum_{t=0}^{T} m_t$ is the sample covariance of $cov(\widetilde{\mathbf{y}}_{t+h|t}(j), \mathbf{y}_{t+h}(k)) - cov(\mathbf{y}_{t+h}(j), \widetilde{\mathbf{y}}_{t+h|t}(k))$. Corollary 1 is from the weak law of large number.

A.6 Proof of Corollary 2

Let $\mathbf{y}' \equiv (\mathbf{y}_{t+h}^{\intercal}, \mathbf{y}_t^{\intercal})^{\intercal}$. Applying Theorem 2 to $\mathbf{y}'$ immediately yields Corollary 2.

A.7 Proof of Proposition 2

$$\widetilde{y}_{t+h|t} = y_t \quad \Longrightarrow \quad (1-\rho\mathbb{L})\left(1-\frac{mpc}{1-\gamma}\mathbb{L}\right) y_t = \gamma \cdot \frac{EIS}{1-\rho\gamma}\epsilon^d_t. \tag{65}$$

A.8 Proof of Proposition 3

Note that future shocks ϵ^d_{t+h} for $h \geq 1$ never affect any covariance statistics in which one of the arguments is a perceived object. Derive the following components

$$\begin{aligned}
std(\pi_{t+4}) &= \sqrt{var(\eta^d_{t+4} - \eta^s_{t+4})} = \sqrt{var(\eta^d_{t+4}) + var(\eta^s_{t+4})} = \sqrt{2(1-\rho^2)^{-1}}, \\
std(g_{t+4}) &= \sqrt{var(\eta^d_{t+4} + \eta^s_{t+4} - \eta^d_t - \eta^s_t)} = \sqrt{var(\eta^d_{t+4} - \eta^d_t) + var(\eta^s_{t+4} - \eta^s_t)} \\
&= \sqrt{var(\eta^d_{t+4}) + var(\eta^d_t) - 2cov(\eta^d_{t+4}, \eta^d_t) + var(\eta^s_{t+4}) + var(\eta^s_t) - 2cov(\eta^s_{t+4}, \eta^s_t)} \\
&= 2\sqrt{(1-\rho^4)(1-\rho^2)^{-1}}, \\
cov(\pi_{t+4}, g_{t+4}) &= cov(\eta^d_{t+4} - \eta^s_{t+4}, \eta^d_{t+4} + \eta^s_{t+4} - \eta^d_t - \eta^s_t) \\
&= cov(\eta^d_{t+4}, \eta^d_{t+4} - \eta^d_t) - cov(\eta^s_{t+4}, \eta^s_{t+4} - \eta^s_t) = 0, \\
cov(\pi_{t+4}, \widetilde{g}_{t+4|t}) &= cov(\eta^d_{t+4} - \eta^s_{t+4}, \widetilde{\eta}^d_{t+4|t} + \widetilde{\eta}^s_{t+4|t} - \widetilde{\eta}^d_{t|t} - \widetilde{\eta}^s_{t|t}) \\
&= cov(\eta^d_{t+4}, \widetilde{\eta}^d_{t+4|t} - \widetilde{\eta}^d_{t|t}) - cov(\eta^s_{t+4}, \widetilde{\eta}^s_{t+4|t} - \widetilde{\eta}^s_{t|t}) \\
&= -\rho^4(1-\rho^4)(1-\rho^2)^{-1}(1-\widetilde{\sigma}_s), \\
cov(\widetilde{\pi}_{t+4|t}, g_{t+4}) &= cov(\widetilde{\eta}^d_{t+4|t} - \widetilde{\kappa} \cdot \widetilde{\eta}^s_{t+4|t}, \eta^d_{t+4} + \eta^s_{t+4} - \eta^d_t - \eta^s_t) \\
&= cov(\widetilde{\eta}^d_{t+4}, \eta^d_{t+4} - \eta^d_t) - cov(\widetilde{\kappa} \cdot \widetilde{\eta}^s_{t+4|t}, \eta^s_{t+4} - \eta^s_t) \\
&= -\rho^4(1-\rho^4)(1-\rho^2)^{-1}(1-\widetilde{\kappa} \cdot \widetilde{\sigma}_s), \\
cov(\widetilde{\pi}_{t+4|t}, \widetilde{g}_{t+4|t}) &= cov(\widetilde{\eta}^d_{t+4|t} - \widetilde{\kappa} \cdot \widetilde{\eta}^s_{t+4|t}, \widetilde{\eta}^d_{t+4|t} + \widetilde{\eta}^s_{t+4|t} - \widetilde{\eta}^d_{t|t} - \widetilde{\eta}^s_{t|t}) \\
&= cov(\widetilde{\eta}^d_{t+4|t}, \widetilde{\eta}^d_{t+4|t} - \widetilde{\eta}^d_{t|t}) - cov(\widetilde{\kappa} \cdot \widetilde{\eta}^s_{t+4|t}, \widetilde{\eta}^s_{t+4|t} - \widetilde{\eta}^s_{t|t}) \\
&= -\rho^4(1-\rho^4)(1-\rho^2)^{-1}(1-\widetilde{\kappa} \cdot \widetilde{\sigma}^2_s).
\end{aligned}$$

Note that $\widetilde{\sigma}_s$ always affects $\widetilde{\eta}^s$, while $\widetilde{\kappa}$ only affects $\widetilde{\eta}^s$ in inflation.

A.9 Proof of Proposition 4

The Kalman filter with potentially misperceived parameters yields

$$\widetilde{\mathbf{y}}_{t+h|t} = \widetilde{\mathbf{\Psi}}\widetilde{\mathbf{x}}_{t+h|t} = \widetilde{\mathbf{\Psi}}\widetilde{\mathbf{\Phi}}^h\widetilde{\mathbf{x}}_{t|t} = \widetilde{\mathbf{\Psi}}\widetilde{\mathbf{\Phi}}^h[\widetilde{\mathcal{K}}\mathbf{x}_t + (\mathbf{I} - \widetilde{\mathcal{K}})\widetilde{\mathbf{x}}_{t|t-1}], \tag{66}$$

in which $\widetilde{\mathcal{K}}$ can be obtained from a static multi-variate signal extraction problem.

A.10 Proof of Proposition 5

Part 1 is due to the spectral representation theorem.

Part 2 is true because when $\widetilde{\mathbf{\Omega}}_{i,\ell} = 0$, forecasters interpret their noisy signal as the realized outcomes so that $\widetilde{\mathbf{y}}_{i,t|t} = \mathbf{y}_t + \mathbf{\Omega}_{i,\ell}\mathbf{e}_{i,\ell,t}$ and hence $\widetilde{\mathbf{y}}_{t|t} = \mathbf{y}_t$.

Part 3 is the special case of Theorem 1.

B Additional Model Details

B.1 An Example of Misspecified Regression for Section 2.2

A natural method is to directly regress forecasts on nowcasts. However, such a method may induce omitted variable bias problem, a special case of model misspecification issue and cause "false positive" testing result. To understand why, consider an example in which $n = 2$, the forecasters observe all data up to the last period, the actual law of motion satisfies $\mathbf{\Psi} = \mathbf{1}$ and

$$\mathbf{x}_{t+1}(1) = \mathbf{\Phi}(1,1)\mathbf{x}_t(1) + \mathbf{\Phi}(1,2)\mathbf{x}_t(2) + \boldsymbol{\epsilon}_{t+1}(1).$$

The perceived version of this actual law of motion in consensus level is

$$\widetilde{\mathbf{x}}_{t+1|t}(1) = \widetilde{\mathbf{\Phi}}(1,1)\widetilde{\mathbf{x}}_{t|t}(1) + \widetilde{\mathbf{\Phi}}(1,2)\widetilde{\mathbf{x}}_{t|t}(2).$$

Consider two regression coefficients below

$$\varphi \equiv \frac{cov(\mathbf{y}_{t+1}(1), \mathbf{y}_t(1))}{var(\mathbf{y}_t(1))} = \mathbf{\Phi}(1,1) + \frac{cov(\mathbf{y}_t(2), \mathbf{y}_t(1))}{var(\mathbf{y}_t(1))}\mathbf{\Phi}(1,2),$$
$$\widetilde{\varphi} \equiv \frac{cov(\widetilde{\mathbf{y}}_{t+1|t}(1), \widetilde{\mathbf{y}}_{t|t}(1))}{var(\widetilde{\mathbf{y}}_{t|t}(1))} = \widetilde{\mathbf{\Phi}}(1,1) + \frac{cov(\widetilde{\mathbf{y}}_{t|t}(2), \widetilde{\mathbf{y}}_{t|t}(1))}{var(\widetilde{\mathbf{y}}_{t|t}(1))}\widetilde{\mathbf{\Phi}}(1,2),$$

in which the two covariance terms satisfy

$$cov(\mathbf{y}_t(2), \mathbf{y}_t(1)) = cov(\mathbf{\Phi}(2,1)\mathbf{y}_{t-1}(1) + \mathbf{\Phi}(2,2)\mathbf{y}_{t-1}(2), \mathbf{\Phi}(1,1)\mathbf{y}_{t-1}(1) + \mathbf{\Phi}(1,2)\mathbf{y}_{t-1}(2)),$$
$$cov(\widetilde{\mathbf{y}}_{t|t}(2), \widetilde{\mathbf{y}}_{t|t}(1)) = cov(\widetilde{\mathbf{\Phi}}(2,1)\mathbf{y}_{t-1}(1) + \widetilde{\mathbf{\Phi}}(2,2)\mathbf{y}_{t-1}(2), \widetilde{\mathbf{\Phi}}(1,1)\mathbf{y}_{t-1}(1) + \widetilde{\mathbf{\Phi}}(1,2)\mathbf{y}_{t-1}(2)),$$

and the two variance terms satisfy

$$var(\mathbf{y}_t(1)) = var(\mathbf{\Phi}(1,1)\mathbf{y}_{t-1}(1) + \mathbf{\Phi}(1,2)\mathbf{y}_{t-1}(2)) + var(\boldsymbol{\epsilon}_t(1)),$$
$$var(\widetilde{\mathbf{y}}_{t|t}(1)) = var(\widetilde{\mathbf{\Phi}}(1,1)\mathbf{y}_{t-1}(1) + \widetilde{\mathbf{\Phi}}(1,2)\mathbf{y}_{t-1}(2)) + \mathcal{K}^2 var(\boldsymbol{\epsilon}_t(1)).$$

When $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}$ and $\mathcal{K}^2 < 1$, it is evident that

$$cov(\widetilde{\mathbf{y}}_{t|t}(2), \widetilde{\mathbf{y}}_{t|t}(1)) = cov(\mathbf{y}_t(2), \mathbf{y}_t(1)),$$
$$var(\widetilde{\mathbf{y}}_{t|t}(1)) = var(\mathbf{y}_t(1)) - (1 - \mathcal{K}^2) var(\boldsymbol{\epsilon}_t(1)) < var(\mathbf{y}_t(1)).$$

As a result, $\widetilde{\varphi} \neq \varphi$ even if $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}$. In other words, directly regressing forecasts on nowcasts to detect misperceived law of motion may induce "false positive" test results when there is perceived information friction. The "self-adjoint method" can avoid this problem because

$$cov(\mathbf{y}_{t+1}(1), \widetilde{\mathbf{y}}_{t|t}(1)) = \mathbf{\Phi}(1,1) cov(\mathbf{y}_t(1), \widetilde{\mathbf{y}}_{t|t}(1)) + \mathbf{\Phi}(1,2) cov(\mathbf{y}_t(2), \widetilde{\mathbf{y}}_{t|t}(1)),$$
$$cov(\widetilde{\mathbf{y}}_{t+1|t}(1), \mathbf{y}_t(1)) = \widetilde{\mathbf{\Phi}}(1,1) cov(\widetilde{\mathbf{y}}_{t|t}(1), \mathbf{y}_t(1)) + \widetilde{\mathbf{\Phi}}(1,2) cov(\widetilde{\mathbf{y}}_{t|t}(2), \mathbf{y}_t(1)),$$

in which $cov(\widetilde{\mathbf{y}}_{t|t}(1), \mathbf{y}_t(1)) = cov(\mathbf{y}_t(1), \widetilde{\mathbf{y}}_{t|t}(1))$ as scalar covariance and

$$\begin{aligned} cov(\mathbf{y}_t(2), \widetilde{\mathbf{y}}_{t|t}(1)) &= cov(\mathbf{\Phi}(2,1)\mathbf{y}_{t-1}(1) + \mathbf{\Phi}(2,2)\mathbf{y}_{t-1}(2), \widetilde{\mathbf{\Phi}}(1,1)\mathbf{y}_{t-1}(1) + \widetilde{\mathbf{\Phi}}(1,2)\mathbf{y}_{t-1}(2)), \\ cov(\widetilde{\mathbf{y}}_{t|t}(2), \mathbf{y}_t(1)) &= cov(\widetilde{\mathbf{\Phi}}(2,1)\mathbf{y}_{t-1}(1) + \widetilde{\mathbf{\Phi}}(2,2)\mathbf{y}_{t-1}(2), \mathbf{\Phi}(1,1)\mathbf{y}_{t-1}(1) + \mathbf{\Phi}(1,2)\mathbf{y}_{t-1}(2)), \end{aligned}$$

so that when $\widetilde{\mathbf{\Phi}} = \mathbf{\Phi}$, $cov(\widetilde{\mathbf{y}}_{t|t}(2), \mathbf{y}_t(1)) = cov(\mathbf{y}_t(2), \widetilde{\mathbf{y}}_{t|t}(1))$ and hence

$$cov(\widetilde{\mathbf{y}}_{t+1|t}(1), \mathbf{y}_t(1)) = cov(\mathbf{y}_{t+1}(1), \widetilde{\mathbf{y}}_{t|t}(1)).$$

In summary, the method that regresses forecasts on nowcasts directly is not valid because $var(\widetilde{\mathbf{y}}_{t|t}(1)) \neq var(\mathbf{y}_t(1))$. In contrast, the "self-adjoint method" is valid because it does not need to compute such variance.

B.2 Micro-foundation of the Model in Section 4.1

Consider the representative households with perfect foresight solving

$$\begin{aligned} &\max_{\{c_{t+h}, b_{t+h}\}} \sum_{\tau=0}^{+\infty} \beta^{\tau} \left[\exp(-\eta^d_{t+h}) u(c_{t+h}) + v(b_{t+h}) \right], \\ &\text{s.t.} \quad c_{t+h} + b_{t+h} \leq R_{t-1+h} b_{t-1+h} + y_{t+h}. \end{aligned}$$

Proposition 6 (Consumption function)**.** *Starting with $b_{t-1} = 0$, the log-linearization solution to households' problem around the steady state with $b_{ss} = 0$ is*

$$\ln(c_t / c_{ss}) = \sum_{\tau=0}^{+\infty} \gamma^h \left[mpc \cdot \ln(y_{t+h}/y_{ss}) + EIS \cdot \gamma \eta^d_{t+h} \right] + o(\{\epsilon^d_{t+h}\}),$$

with $\gamma \equiv \beta R_{ss}(1 - mpc)$, $EIS \equiv \frac{u'(c_{ss})}{-u''(c_{ss})c_{ss}}$ and steady state conditions

$$\beta R_{ss} = 1 - \frac{v'(0)}{u'(c_{ss})} < 1, \quad mpc = \frac{\sqrt{\left(1 - \beta R_{ss}^2 + \frac{v''(0)}{u''(c_{ss})}\right)^2 + 4\beta R_{ss}^2 \frac{v''(0)}{u''(c_{ss})}} - \left(1 - \beta R_{ss}^2 + \frac{v''(0)}{u''(c_{ss})}\right)}{2\beta R_{ss}^2}.$$

C Transforming MSC Data

Assumption 1 (Reporting rule)**.** *At each period t, the subjective expectations of the respondents follow a Gaussian distribution with mean μ_t and variance σ_t^2. For the unconditional mean $\mathbb{E}\mu_t$ and the width of inattentive interval ψ, all respondents answer "better" if their expectations are above $\mathbb{E}\mu_t + \frac{1}{2}\psi$, and "worse" if below $\mathbb{E}\mu_t - \frac{1}{2}\psi$.*

Assumption 1 helps transform the data by restricting the forecasters to share a common reporting rule characterized by a single parameter ψ. Using $G(\cdot)$ to denote the cumulative distribution function of a standard now distribution, we have

$$\%BETTER_t = 1 - G\left(\frac{\mathbb{E}\mu_t + \frac{1}{2}\psi - \mu_t}{\sigma_t}\right), \quad \%WORSE_t = G\left(\frac{\mathbb{E}\mu_t - \frac{1}{2}\psi - \mu_t}{\sigma_t}\right). \tag{67}$$

Inverting out μ_t and σ_t yields

$$\mu_t - \mathbb{E}\mu_t = -\frac{\psi}{2} \cdot \frac{G^{-1}(1 - \%BETTER_t) + G^{-1}(\%WORSE_t)}{G^{-1}(1 - \%BETTER_t) - G^{-1}(\%WORSE_t)}, \tag{68}$$

$$\sigma_t = \frac{\psi}{G^{-1}(1 - \%BETTER_t) - G^{-1}(\%WORSE_t)}, \tag{69}$$

in which $\mu_t - \mathbb{E}\mu_t$ is the transformed demeaned consensus expectations.

Assumption 2 (Common rule)**.** *Nowcasts and forecasts share the same ψ in reporting rules.*

Assumption 2 implies that although the transformation from categorical to numerical data is scaled up by the parameter ψ, the method of Theorem 2 is not affected by ψ because both sides of the null hypothesis are scaled up by the same scaling factor ψ. Therefore, we have Proposition 7. The transformed data is depicted in Figure 1.

Proposition 7 (Irrelevant scaling)**.** *z^{fore}/z^{now} is not affected by ψ.*

Proposition 7 makes sure that the main test results for misperceived law of motion in Section 4.1 is not sensitive to how the scaling parameter ψ is chosen if both Assumption 1 and Assumption 2 hold.